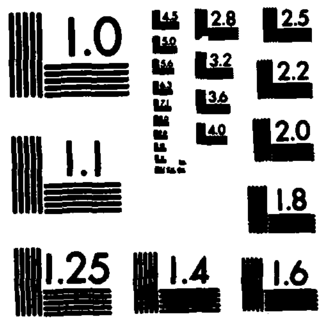
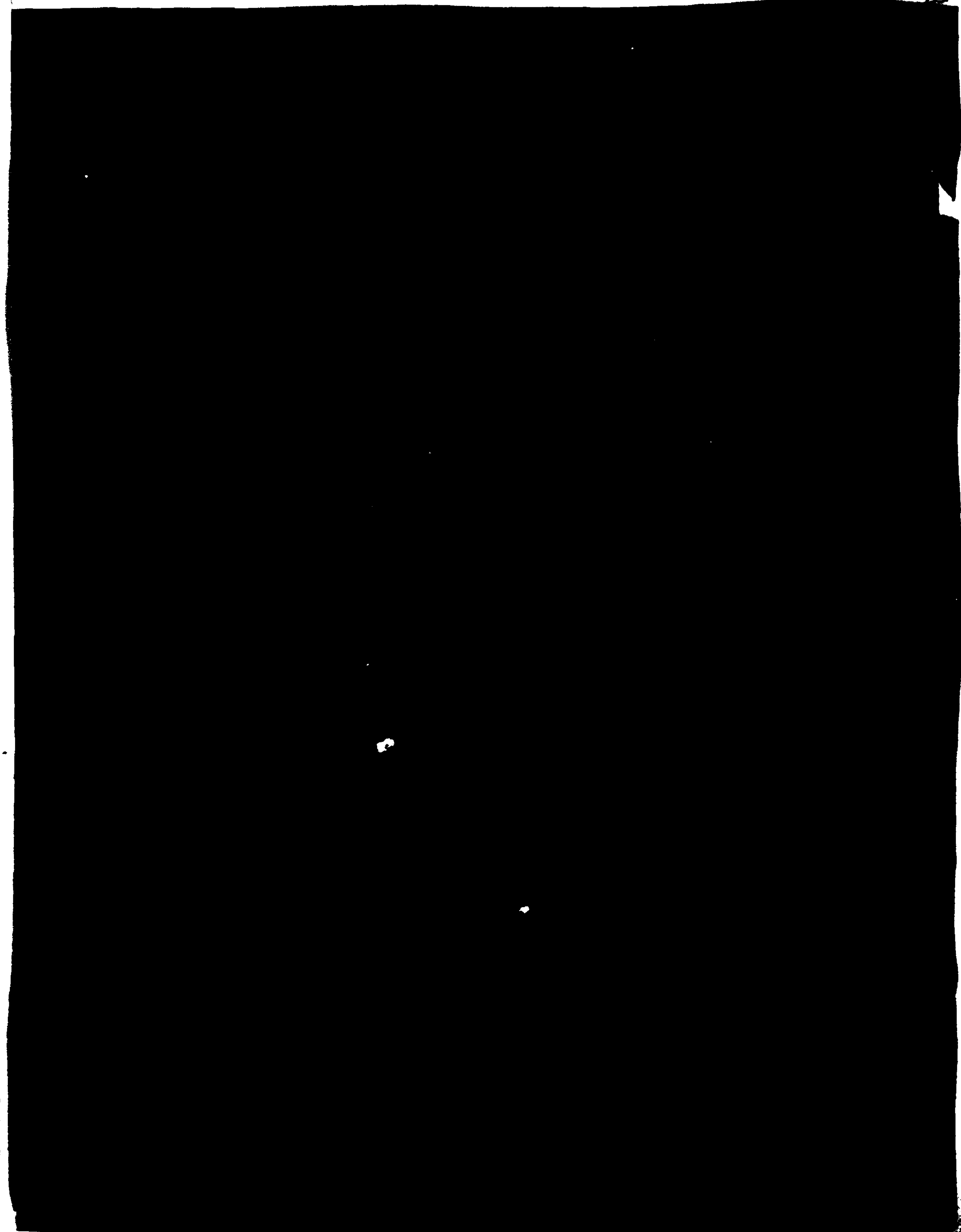
AD-A121 433 ON THE MATHEMATICAL THEORY OF THE MOTION OF FLOATING BODIES - AN UPDATE(U) DAVID W TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER BETHESDA MD 1/1

UNCLASSIFIED R E KLEINMAN OCT 82 DTNSRDC-82/074 F/G 12/1 NL

END
DATE
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER DTNSRDC-82/074	2. GOVT ACCESSION NO. AD-A121 433	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE MATHEMATICAL THEORY OF THE MOTION OF FLOATING BODIES - AN UPDATE		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ralph E. Kleinman		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS David W. Taylor Naval Ship Research and Development Center Bethesda, Maryland 20084		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (See reverse side)
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research (Code 432) Arlington, Virginia 22217		12. REPORT DATE October 1982
		13. NUMBER OF PAGES 67
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

DTIC ELECTE NOV 12 1982 S D A

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Ship Waves
Irregular Frequencies

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The generation of surface waves in a fluid caused by a partially submerged body is modeled mathematically as a boundary value problem for the Laplacian with Neumann data on the bottom of the fluid container and on the body, a linearized free surface condition and a radiation condition at infinity. Fritz John (Comm. Pure Appl. Math., III, 1950) showed that the problem had a unique solution only under rather restrictive assumptions on the

(Continued on reverse side)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(Block 10)

Program Element 61153N
Task Area RR0140302
Work Unit 1542-018

(Block 20 continued)

body geometry and reformulated the problem as an integral equation which was not uniquely solvable at certain irregular frequencies. In the present work, uniqueness is established for more general geometries, allowing corners and nonnormal intersections of the body with the free surface. Also presented are two methods of modifying the integral equation so that it is uniquely solvable for all frequencies. One method involves introducing an additional integral on the waterplane, while in the second method, an additional integral term is added to the equation which remains an equation only over the submerged portion of the body.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

The generation of surface waves in a fluid caused by a partially submerged body is modeled mathematically as a boundary value problem for the Laplacian with Neumann data on the bottom of the fluid container and on the body, a linearized free surface condition and a radiation condition at infinity. Fritz John (Comm. Pure Appl. Math., III, 1950) showed that the problem had a unique solution only under rather restrictive assumptions on the body geometry and reformulated the problem as an integral equation which was not uniquely solvable at certain irregular frequencies. In the present work, uniqueness is established for more general geometries, allowing corners and nonnormal intersections of the body with the free surface. Also presented are two methods of modifying the integral equation so that it is uniquely solvable for all frequencies. One method involves introducing an additional integral on the waterplane, while in the second method, an additional integral term is added to the equation which remains an equation only over the submerged portion of the body.

ADMINISTRATIVE INFORMATION

The research reported here was proposed by the Numerical Ship Hydrodynamics Program at the David W. Taylor Naval Ship Research and Development Center (DTNSRDC). This program is jointly supported by the Office of Naval Research under Program Element 61153N, Task Area RR0140302, and by the Independent Research Program at DTNSRDC under Program Element 61152N, Task Area ZR0230101, using Work Units 1552-018 and 1843-015. The research was performed at DTNSRDC by the author during sabbatical leave from the University of Delaware under the Intergovernmental Personnel Act of 1970.

INTRODUCTION

It has been thirty years since Fritz John published a pair of papers with a similar title[1*,2] which still underlie present thinking. In Reference 2, which must be regarded as a tour de force of classical applied mathematics, John analyzed simple harmonic motion of the fluid in which an impenetrable body is partially immersed. He formulated the problem mathematically as a boundary value problem for Laplace's equation with appropriate boundary conditions on the body, the free surface, the bottom of the fluid container, and the radiation condition. Under certain restrictions on the body shape, he proved the existence of a unique solution and formulated

*A complete listing of references is given on page 57.

an integral equation for the velocity potential evaluated on the body surface. This equation is not uniquely solvable at a set of irregular frequencies as John pointed out and this has plagued researchers in this area ever since.

John based his work on classical potential theory even though the physical domain had a corner at the intersection of the body and the free surface. Thus, he looked for classical solutions, continuously differentiable up to and including the boundary, although he did not show that the solution, the existence of which he proved, actually satisfied this property. This is not surprising in the light of subsequent work[3] which shows, in fact, that such smooth solutions do not exist. However, by employing the fundamental work on potential theory for irregular domains, developed by Burago et al.,[4] Král,[5] and Wendland,[6] it is possible to formulate the problem in a more appropriate mathematical setting and to show the existence of a unique solution under less restrictive assumptions on the obstacle shape. This is done in the present report where uniqueness and existence are established for piecewise smooth bodies. If we permit the wave number to have a nonzero imaginary part, no further restrictions are required, whereas we do restrict the bodies to those whose projection onto the free surface coincides with the waterplane area if the wave number has real values. In either case, we relax the conditions John imposed and permit nonnormal intersections of the body with the free surface as well as bodies with corners.

We also present two methods of resolving the problem of irregular frequencies. One is essentially a simplified version of the method proposed by Chang and Pien[7] in which the obstacle surface is augmented by the waterplane to obtain an integral equation on a closed surface soluble at all frequencies. The second method is modeled after a treatment of exterior problems for the Helmholtz equation where similar problems occur; Burton and Miller,[8] Kleinman and Roach,[9] and Angell and Kleinman.[10] Here the integral operator is augmented rather than its domain and range which remain the submerged portion of the body. An alternate method for removing the irregular frequencies has been given by Ursell,[11] using a modified Green's function as proposed by Jones,[12] to handle a similar problem in acoustic scattering.

NOTATION AND FORMULATION

As illustrated in Figures 1 and 2, the geometry of our problem is described in the following manner. Let D_- be a connected bounded open set in three-dimensional space, $\mathbb{R}^3$, whose boundary consists of $\bar{C}_w$, a closed bounded connected region in the

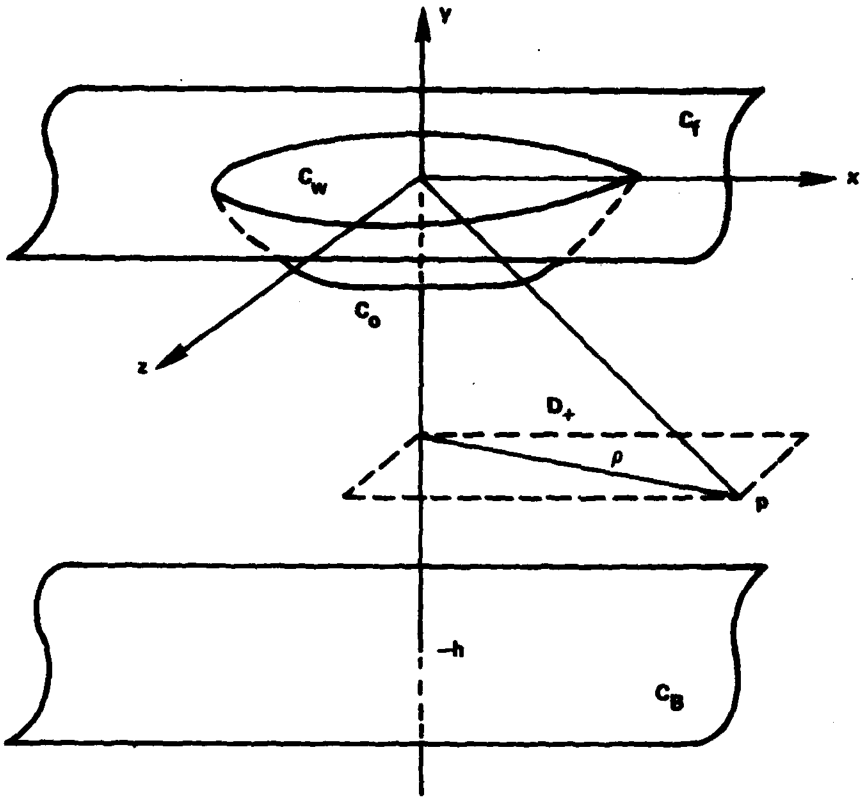


Figure 1 - Floating Body - Three-Dimensional View

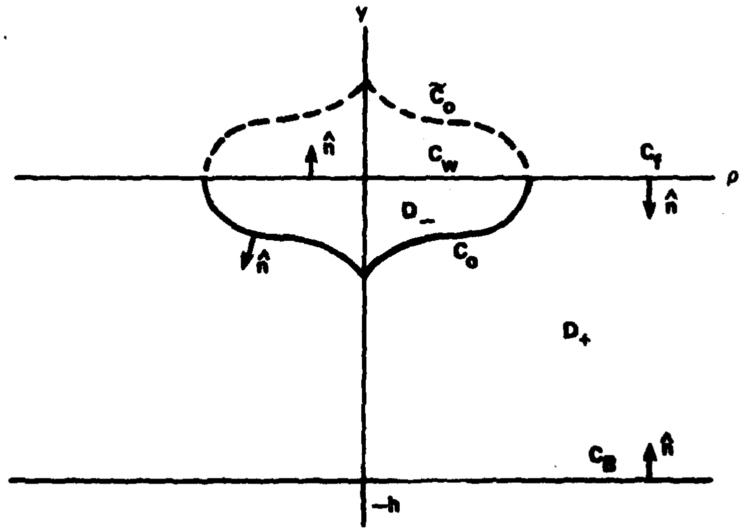


Figure 2 - Floating Body - Section in (ρ, y) Plane

x-z plane and C_o, a piecewise smooth surface in the lower half space, $y < 0$, in the following sense. The boundary $\partial D_- = \bar{C}_w \cup C_o$ is to be piecewise Liapunov, that is, composed of a finite number of segments each of which lies on a Liapunov surface (on which there is a Hölder continuous normal); see, e.g., Günter[13] for a precise definition of Liapunov surfaces. Moreover, ∂D_- satisfies a two-sided cone condition; i.e., there exist α, $h > 0$, such that each point of ∂D_- may be taken as the vertex of two cones of height h and vertex angle α, one lying entirely in D_- and one in the complement of the closure of D_-. A rectangular coordinate system is oriented with the origin in C_w and D_- lying below the $y = 0$ plane. Let C_f, the free surface, be the closure of the complement of C_w in the $y = 0$ plane, and let C_B be the plane $y = -h$. Denote by D_+ the unbounded domain bounded by C_f, C_o, and C_B. Also let $\tilde{C}_o$ be the reflection of C_o in the x-z plane and $\tilde{D}_-$ be the reflection of D_-. Unless otherwise noted, the normal $\hat{n}$ points into D_+ from C_o, C_f, C_B, and into $\tilde{D}_-$ from C_w. We denote points (vectors) by $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ with cylindrical coordinates $p = (\rho_p, \theta_p, y_p)$ and the subscripts will be omitted if there is no danger of confusion.

We formulate the floating body problem as follows:

Find a function

$$\phi(p) \in C^2(D_+) \cap C^o(D_+ \cup \partial D_+) \tag{1a}$$

where $C^m(\Omega)$ is the space of functions with m continuous derivatives on Ω, such that,

$$\nabla^2\phi = 0 \qquad p \in D_+ \tag{1b}$$

$$\frac{\partial}{\partial n}\phi + k\phi = -\frac{\partial\phi}{\partial y} + k\phi \qquad p \in C_f \tag{1c}$$

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial y} = 0 \qquad p \in C_B \tag{1d}$$

$$\frac{\partial\phi}{\partial n} = V(p) \qquad p \in C_o \tag{1e}$$

where $V(p)$ is integrable on C_o,

$$\frac{\partial\phi}{\partial\rho} - ik_o\phi = o\left(\frac{1}{\rho^{1/2}}\right) \tag{1f}$$

where k_o is the root with largest real part of the equation

$$k_n \sinh k_n h = k \cosh k_n h \tag{2}$$

Some properties of k_n are listed in Appendix A; here we only remark that there always is such a k_o with a real part that is strictly positive.

Finally we add a condition on $\partial\phi/\partial n$ which will enable us to employ Green's theorem, namely, that $\partial\phi/\partial n$ defines a distribution in $C^{o*}(\partial D_+)$ (the space of functions of bounded variation) in the sense of boundary flow (see Burago, Maz'ja and Sapoznikhova,[4] Wendland,[6] and Kleinman and Wendland[14] for a more detailed discussion of this concept). Roughly this means that

$$\lim_{\partial D_m \to \partial D_+} \int_{\partial D_m} u \frac{\partial\phi}{\partial n}\, ds$$

defines a bounded linear functional on $C^o(\partial D_+)$ (continuous functions on ∂D_+) where $u \in C_o^\infty(\mathbb{R}^3)$ (infinitely differentiable functions with compact support in $\mathbb{R}^3$) and ∂D_m is a family of smooth surfaces in D_+ converging to ∂D_+. We note that, if $(\partial\phi/\partial n) \in L^1_{loc}(\partial D_+)$, it will have boundary flow. Alternatively, we could require that $\phi(p) \in H^1_{loc}(D_+)$, that is, $\int_\Omega \{|\phi|^2 + |\nabla\phi|^2\} d\tau < \infty$ for every bounded subdomain $\Omega \subset D_+$. This would also enable us to employ Green's theorem. Kleinman and Wendland[14] have shown that, if ϕ has boundary flow, then it is in $H^1_{loc}(D_+)$. The given data V will be interpreted as defining a boundary flow which it will do if, for example, $V \in L^1(C_o)$. Here, $L^1(\Omega)$ is the space of (Lebesgue) integrable functions on Ω; $H^1(\Omega)$ is the space of functions which, together with their first derivatives, are in $L^1(\Omega)$; and $L^1_{loc}(\Omega)$ is the space of functions integrable over every bounded subdomain of Ω.

UNIQUENESS

The first important property we establish concerns conditions under which the problem, Equations (1a)-(1f), has a unique solution. We remark that John proved uniqueness by requiring that C_o was smooth (C^2) and that rays perpendicular from C_B intersect C_o at most once. In fact, he requires C_o (or $C_o \cup \tilde{C}_o$) to be convex. Here we will establish two versions of the uniqueness proof. In one we require the waves to be dispersive, that is, Im $k > 0$, in which case uniqueness is established for the piecewise smooth boundaries described earlier. The second deals with nondispersive waves, Im $k = 0$, and requires an additional assumption that vertical rays from C_f (not C_B as in John's proof) do not contain points in D_-. In either case, we make no convexity requirements and permit nonnormal intersections of C_o and C_f. We note that the assumption of a small imaginary part for k has ample precedent (e.g., Lighthill[15] and Noblesse.[16] Also, it should be noted that Lenoir and Martin[17] have presented a uniqueness proof for real k in the case of infinite depth which requires no geometric restrictions. Their method, however, does not appear to be immediately extendable to finite depth. In fact, Ursell[18] has shown that their proof is incorrect.

Uniqueness Theorem:

If C_o is piecewise smooth and Im $k > 0$, the homogeneous floating body problem has only the trival solution; that is, if

$$\nabla^2\phi = 0,\ p \in D_+ \qquad \frac{\partial\phi}{\partial n} = 0,\ p \in C_o$$

$$\frac{\partial\phi}{\partial y} - k\phi = 0,\ p \in C_f \qquad \frac{\partial\phi}{\partial\rho} - ik_o\phi = o\left(\frac{1}{\rho^{1/2}}\right) \tag{3}$$

$$\frac{\partial\phi}{\partial y} = 0,\ p \in C_B \qquad \phi \in C^2(D_+) \cap C^o(D_+ \cup \partial D_+)$$

and $\partial\phi/\partial n$ exists as a boundary flow even though it is not defined at corners, then $\phi \equiv 0$. The theorem remains true if Im $k = 0$, and vertical rays from C_f do not intersect D_-.

Proof:

Assume ϕ satisfies Equation (3). Since ϕ is continuous in $C^{o}\ (D_{+} \cup \partial D_{+})$ and has a normal derivative in the sense of boundary flow, we may still apply the divergence theorem (Burago and Maz'ja)[19] to $\phi\nabla\bar{\phi} - \bar{\phi}\nabla\phi$ in $D_{+} \cup D_R$, where $D_R = \{p|\rho<R\}$ is a cylinder of radius R; and $\bar{\phi}$ denotes the complex conjugate, thus obtaining

$$\int_{D_{+}\cap D_R} \{\phi\nabla^2\bar{\phi}-\bar{\phi}\nabla^2\phi\}d\tau = -\int_{C_f \cup C_o \cup C_B \cup \partial D_R} \left\{\phi\,\frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}\,\frac{\partial\phi}{\partial n}\right\} ds \tag{4}$$

where $\hat{n}$ points into D_{+}. Because ϕ satisfies Laplace's equation, the integral on the left vanishes and the boundary conditions cause the integrals over C_o and C_B to vanish as well. Employing the boundary condition on C_f and the fact that on ∂D_R, $\partial/\partial n = -\partial/\partial\rho$, we obtain

$$\int_{C_f} (\bar{k}-k)\ |\phi|^2\ ds + \int_{\partial D_R} \left(\phi\,\frac{\partial\bar{\phi}}{\partial\rho} - \bar{\phi}\,\frac{\partial\phi}{\partial\rho}\right) ds = 0 \tag{5}$$

The radiation condition implies that on ∂D_R

$$\begin{aligned}\left|\frac{\partial\phi}{\partial\rho} - ik_o\phi\right|^2 &= \left(\frac{\partial\phi}{\partial\rho} - ik_o\phi\right)\left(\frac{\partial\bar{\phi}}{\partial\rho} + i\bar{k}_o\bar{\phi}\right)\\ &= \left|\frac{\partial\phi}{\partial\rho}\right|^2 + |k_o\phi|^2 + \operatorname{Im} k_o\left(\phi\,\frac{\partial\bar{\phi}}{\partial\rho} + \bar{\phi}\,\frac{\partial\phi}{\partial\rho}\right) + i\operatorname{Re}k_o\left(\bar{\phi}\,\frac{\partial\phi}{\partial\rho} - \phi\,\frac{\partial\bar{\phi}}{\partial\rho}\right) \qquad (6)\\ &= o\left(\frac{1}{\rho}\right) = o\left(\frac{1}{R}\right)\end{aligned}$$

hence,

$$\phi\,\frac{\partial\bar{\phi}}{\partial\rho} - \bar{\phi}\,\frac{\partial\phi}{\partial\rho} = \frac{1}{i\operatorname{Re}k_o}\left\{\left|\frac{\partial\phi}{\partial\rho}\right|^2 + |k_o\phi|^2 + \operatorname{Im} k_o\left(\phi\,\frac{\partial\bar{\phi}}{\partial\rho} + \bar{\phi}\,\frac{\partial\phi}{\partial\rho}\right)\right\} + o\left(\frac{1}{\rho}\right) \tag{7}$$

Substituting in Equation (5) we obtain

$$\int_{C_f} (\bar{k}-k)\ |\phi|^2\ ds + \frac{1}{iRek_o}\int_{\partial D_R} \left\{\left|\frac{\partial\phi}{\partial\rho}\right|^2 + |k_o\phi|^2 + Im\ k_o\left(\phi\ \frac{\partial\bar{\phi}}{\partial n} + \bar{\phi}\ \frac{\partial\phi}{\partial\rho}\right)\right\} ds = o(1)$$

$$\text{as } R \to \infty \qquad (8)$$

or

$$2\ Im\ k \int_{C_f} |\phi|^2\ ds + \frac{1}{Rek_o}\int_{\partial D_R} \left\{\left|\frac{\partial\phi}{\partial\rho}\right|^2 + |k_o\phi|^2 + Im\ k_o\left(\phi\ \frac{\partial\bar{\phi}}{\partial n} + \bar{\phi}\ \frac{\partial\phi}{\partial n}\right)\right\} ds = o(1)$$

$$\text{as } R \to \infty \qquad (9)$$

From property 4 in Appendix A we know that if $k \neq 0$, then $Rek_o > 0$. Equation (9) may be written as

$$2\ Im\ k \int_{C_f} |\phi|^2\ ds + \frac{1}{Rek_o}\int_{\partial D_R} \left\{(Rek_o)^2\ |\phi|^2 + \left(Im\ k_o\ Re\phi + Re\ \frac{\partial\phi}{\partial\rho}\right)^2\right.$$

$$\left. + \left(Im\ k_o\ Im\phi + Im\ \frac{\partial\phi}{\partial\rho}\right)^2\right\} ds = o(1) \qquad (10)$$

Since $Im\ k \geq 0$, then all the terms on the left are nonnegative, and it follows that individually they vanish as $R \to \infty$. Thus,

$$\int_{\partial D_R} |\phi|^2\ ds = o(1) \Rightarrow \phi = o\left(\frac{1}{\rho^{1/2}}\right) \qquad (11)$$

which, together with the remaining relations, ensures that

$$\frac{\partial\phi}{\partial\rho} = o\left(\frac{1}{\rho^{1/2}}\right) \qquad (12)$$

and

$$\operatorname{Im} k \int_{C_f} |\phi|^2 \, ds = o(1) \tag{13}$$

But the last expression is independent of R, hence

$$\operatorname{Im} k \int_{C_f} |\phi|^2 \, ds = 0 \tag{14}$$

Equations (11) and (12) imply that $\bar{\phi}\ \partial\phi/\partial n \to 0$ on ∂D_R as $R \to \infty$, hence we may apply the divergence theorem to $\bar{\phi}\nabla\phi$ in D_+ , obtaining

$$-\int_{C_o \cup C_f \cup C_B} \bar{\phi} \frac{\partial\phi}{\partial n} \, ds = \int_{D_+} |\nabla\phi|^2 \, d\tau \tag{15}$$

which, with the boundary conditions, reduces to

$$k \int_{C_f} |\phi|^2 \, ds = \int_{D_+} |\nabla\phi|^2 \, d\tau \tag{16}$$

If $\operatorname{Im} k > 0$, then Equation (14) implies that $\int_{C_f} |\phi|^2 \, ds = 0$ and, therefore,

$$\int_{D_+} |\nabla\phi|^2 \, d\tau = 0$$

Thus ϕ must be constant and, by Equation (14), the constant must be zero and the theorem is proven. If $\operatorname{Im} k = 0$, Equation (14) gives no information and Equation (16), although still valid does not imply that $\phi = 0$ since k has the "wrong" sign.

To prove uniqueness, when $\mathrm{Im}\, k = 0$ we follow John's argument and impose the additional restriction that every vertical ray from the free surface contains no points of D_-; that is, if $(x,0,z) \in C_f$ then $(x,y,z) \notin D_-$, for $-h \leq y \leq 0$. This restriction means that the projection of D_- on the $y = 0$ plane is C_w, the waterplane, as illustrated in Figure 3a. We introduce additional

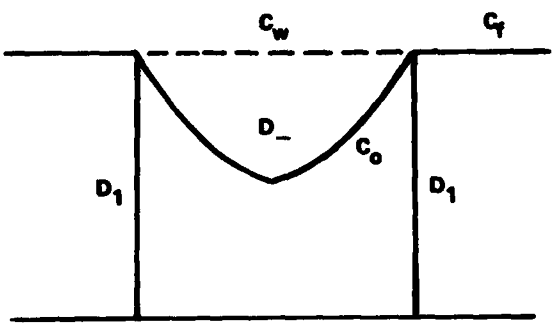


Figure 3a - Permissible Configuration

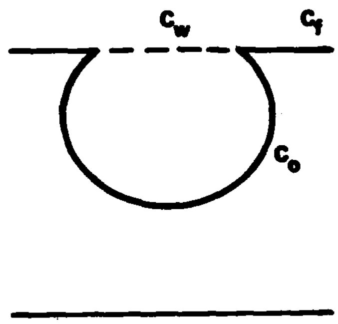


Figure 3b - Impermissible Configuration

Figure 3 - Restricted Geometry for $\mathrm{Im}\, k = 0$

notation setting D_1 to be that portion of D_+ that may be reached by vertical rays from C_f and, following John, we have the representation

$$\phi = \sum_{n=0}^{\infty} a_n(x,z) \cosh k_n(y+h) \qquad \text{for } p \in D_1 \tag{17}$$

where k_n are the roots of Equation (2) with nonnegative imaginary part. Using Equations (17) and (2) we may show, by elementary integration, that

$$\int_{-h}^{0} \phi(x,y,z) \cosh k_o(y+h)\, dy = \frac{h}{2}\, a_o(x,z) \left(1+ \frac{\sinh 2k_o h}{2k_o h}\right) \tag{18}$$

With the Schwarz inequality we find

$$\left|\frac{h}{2}\, a_o \left(1+\frac{\sinh\, 2k_o h}{2k_o h}\right)\right|^2 \le \int_{-h}^{0} |\phi|^2\, dy \int_{-h}^{0} \cosh^2 k_o(y+h)\, dy \tag{19}$$

therefore, there is a constant C $\left(= \dfrac{4}{\dfrac{\sinh\, 2k_o h}{2k_o} + h}\right)$ such that

$$|a_o(x,z)|^2 \le C \int_{-h}^{0} |\phi|^2\, dy \tag{20}$$

Integrating Equation (20) around a circle of radius R we have

$$\int_0^{2\pi} R|a_o|^2\, d\theta \le C \int_{\partial D_R} |\phi|^2\, ds \tag{21}$$

and letting $R \to \infty$, using Equation (11), it follows that

$$\lim_{R\to\infty} \int_0^{2\pi} R|a_o|^2\, d\theta = 0 \tag{22}$$

Since

$$a_o = \sum_{n=-\infty}^{\infty} \beta_n H_{|n|}^{(1)} (k_o\rho)\, e^{in\theta} \tag{23}$$

where $H_{|n|}^{(1)}$ are Hankel functions,

$$\lim_{R\to\infty}\int_0^{2\pi} R|a_o|^2\, d\theta = \lim_{R\to\infty}\int_0^{2\pi} R\left|\sum_{n=-\infty}^{\infty}\beta_n H_{|n|}^{(1)}(k_oR)\, e^{in\theta}\right|^2 d\theta$$

$$= \lim_{R\to\infty} 2\pi R \sum_{n=-\infty}^{\infty} |\beta_n|^2\, |H_{|n|}^{(1)}(k_oR)|^2 \tag{24}$$

$$= 0$$

from which we infer that

$$\beta_n = 0 \qquad \text{for every } n \tag{25}$$

hence,

$$a_o = 0$$

and

$$\int_{-h}^{0} \phi \cosh k_o(y+h)\, dy = 0 \tag{26}$$

Integration by parts leads to

$$\phi(x,0,z) \sinh k_oh = \int_{-h}^{0} \phi_y \sinh k_o(y+h)\, dy \tag{27}$$

hence,

$$|\phi(x,0,z) \sinh k_oh|^2 \le \int_{-h}^{0} |\phi_y|^2\, dy \int_{-h}^{0} \sinh^2 k_o(y+h)\, dy \tag{28}$$

Since, from Equation (2),

$$\int_{-h}^{0} \sinh^2 k_o(y+h)\, dy = \frac{\sinh k_o h \cosh k_o h}{2k_o} - \frac{h}{2} = \frac{\sinh^2 k_o h}{2k} - \frac{h}{2} \tag{29}$$

we have, with Equation (28),

$$2k\, |\phi(x,0,z)|^2 \leq \int_{-h}^{0} |\phi_y|^2\, dy \leq \int_{-h}^{0} |\nabla\phi|^2\, dy \tag{30}$$

Integrating over C_f we obtain

$$2k \int_{C_f} |\phi|^2\, ds \leq \int_{D_1} |\nabla\phi|^2\, d\tau \tag{31}$$

and, with Equation (16),

$$2 \int_{D_+} |\nabla\phi|^2\, d\tau \leq \int_{D_1} |\nabla\phi|^2\, d\tau \tag{32}$$

or

$$\int_{D_+} |\nabla\phi|^2\, d\tau + \int_{D_+ \setminus D_1} |\nabla\phi|^2\, d\tau \leq 0 \tag{33}$$

from which we infer, since both terms on the left are nonnegative, that $\nabla\phi = 0$ in D_-, hence ϕ = constant. But, then $\partial\phi/\partial n = 0$ and the free surface condition implies that $\phi = 0$ completing the proof. We note that the proof would be essentially unchanged if, instead of $\partial\phi/\partial n = 0$, we had $\phi = 0$ on C_o.

INTEGRAL EQUATION FORMULATION

Again we follow John and introduce the Green's function for the limiting case when D_- is empty and C_f is the entire y = 0 plane. Thus, define

$$\gamma(p,q) := -i \sum_{n=0}^{\infty} \frac{(k_n^2-k^2)H_o^{(1)}(k_n\widetilde{R})}{hk_n^2-hk^2+k} \cosh k_n(y_p+h) \cosh k_n(y_q+h)$$

$$:= -\frac{1}{\pi}\int_0^{\infty} J_o(\mu\widetilde{R}) \cosh \mu(y< + h) \frac{\mu \cosh \mu y> + k \sinh \mu y>}{\mu \sinh \mu h - k \cosh \mu h}\, d\mu \qquad (34)$$

where $y> = \max\{y_p,y_q\} \leq 0$ and $y< = \min\{y_p,y_q\} \geq -h$;

$$\widetilde{R} = \sqrt{(x_p-x_q)^2+(z_p-z_q)^2} \qquad (35)$$

Here k_n are the roots of Equation (2) with nonnegative real and imaginary parts, and the contour passes below the zeros, $\mu = k_n$, of the denominator. *It is not difficult* to see that γ satisfies the radiation condition and boundary conditions,

$$\frac{\partial\gamma}{\partial y} = 0,\ y = -h \quad \text{and} \quad \frac{\partial\gamma}{\partial y} - k\gamma = 0,\ y = 0 \qquad (36)$$

John has also shown that the singular behavior of γ is given by

$$\gamma = -\frac{1}{2\pi|p-q|} - \frac{1}{2\pi|p-\widetilde{q}|} - \frac{1}{2\pi|p-q_1|} + H(p,q) \qquad (37)$$

where $H = O(\ln|p-\widetilde{q}|)\ O(\ln|p-q_1|)$

$\widetilde{q} = (x_q,-y_q,z_q)$

$q_1 = (x_q,-y_q-2h,z_q)$

We now state Green's theorem for solutions for the floating body problem using the Green's function defined in Equation (34).

If

$$\nabla^2\phi = 0, \ p \in D_+$$

$$\frac{\partial\phi}{\partial\rho} - ik_o\phi = o\left(\frac{1}{\rho}\right)$$

$$\frac{\partial\phi}{\partial y} = 0, \ y = -h$$

$$\frac{\partial\phi}{\partial y} - k\phi = 0, \ p \in C_f$$

$$\text{Im } k \geq 0$$

and $\partial\phi/\partial y$ exists at least in the sense of boundary flow, then

$$\int_{C_o} \left\{\phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} - \gamma(p,q) \frac{\partial\phi}{\partial n_q}\right\} ds_q = -\alpha(p)\,\phi(p) \tag{38}$$

where $\hat{n}_q$ points into D_+ and

$$\alpha(p) := \lim_{\varepsilon\to 0} \int_{[\partial B_\varepsilon(p)]\cap D_+} \frac{\partial\gamma}{\partial n_q} ds_q = \begin{cases} 2, & p \in D_+ \cup C_f' \cup C_B \\ 1, & p \in \text{smooth points of } C_o \\ 0, & p \in D_- \cup C_w' \end{cases} \tag{39}$$

The fact that $\alpha(p) = 2$ on C_f and C_B rather than 1 is a consequence of the form of γ, Equation (37). Here $B_\varepsilon(p)$ denotes a sphere of radius ε with center at p, the normal points away from $B_\varepsilon(p)$ and C_f' and C_w' are primed to denote that the boundary points between C_f and C_w are not included. It should be noted that $\alpha(p)$ is defined at non-smooth as well as smooth points and is a measure of a normalized solid angle, although the explicit evaluation in Equation (39) holds only at smooth points.

We also state a version of Green's theorem for solutions of Laplace's equation in D_-: If $\nabla^2\tilde{\phi} = 0$, $p \in D_-$ and $\partial\tilde{\phi}/\partial n$ exists at least as an interior boundary flow (approximating surfaces lie in D_-), then

$$\int_{C_o \cup C_w} \left\{ \tilde{\phi}(q) \frac{\partial \gamma(p,q)}{\partial n_q} - \gamma \frac{\partial \tilde{\phi}}{\partial n_q} \right\} ds_q = [2-\alpha(p)]\, \tilde{\phi}(p) \qquad (40)$$

where $\hat{n}_q$ points away from D_-.

Since $\partial\gamma/\partial n_q = k\gamma$ on C_w, this may be written

$$\int_{C_o} \left\{ \tilde{\phi}(q) \frac{\partial \gamma}{\partial n_q} - \gamma \frac{\partial \tilde{\phi}}{\partial n_q} \right\} ds_q - \int_{C_w} \gamma(p,q) \left(\frac{\partial \tilde{\phi}}{\partial y_q} - k\tilde{\phi} \right) ds_q = [2-\alpha(p)]\, \tilde{\phi}(p) \qquad (41)$$

In particular, if $\tilde{\phi} = 1$, we get a form of Gauss' integral

$$\int_{C_o} \frac{\partial \gamma}{\partial n_q} ds_q + k \int_{C_w} \gamma \, ds_q = 2 - \alpha(p) \qquad (42)$$

It should be kept in mind that both Equations (38) and (42) hold for points p in the slab $-h \leq y_p \leq 0$ since, as yet, we have *not defined* γ *outside of this region.* With the boundary condition, Equation (1e), and Equation (38), we have the boundary integral equation

$$\alpha(p)\, \phi(p) + \int_{C_o} \phi(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in C_o \qquad (43)$$

where $\alpha(p) = 1$ at all smooth points of C_o. Alternatively, we could multiply Equation (42) by $\phi(p)$ and add it to Equation (38), and use Equation (1e), to obtain

$$2\, \phi(p) + \int_{C_o} [\phi(q) - \phi(p)] \frac{\partial \gamma(p,q)}{\partial n_q} ds_q - k\phi(p) \int_{C_w} \gamma(p,q)\, ds_q$$

$$= \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in D_+ \cup \partial D_+ \qquad (44)$$

This is a form derived by Noblesse[16] and is the analog of the formulation for the Helmholtz equation derived by Ahner and Kleinman.[20] Observe that the representation does not change form when p is on the boundary of D_+ even if p is a corner point. If we restrict p to lie on C_o, both Equations (43) and (44) are integral equations for the unknown values of ϕ on C_o. As John showed, there exist irregular values of k, that is, those values for which

$$\alpha(p)\, u(p) + \int_{C_o} u(q) \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q = 0 \qquad \text{for } p \in C_o \qquad (45)$$

has nontrivial solutions. We note that if u is a solution of Equation (45), it is also a solution of the homogeneous form of Equation (44). Note also that since $\alpha(p) = 1$ almost everywhere on C_o, Equation (45) may also be written

$$\alpha(p)\, u(p) + \int_{C_o} \alpha(q)\, u(q) \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q = 0 \qquad \text{for } p \in C_o \qquad (46)$$

and so, even if C_o is only piecewise smooth, the irregular frequencies are those for which there exist piecewise continuous nontrivial solutions of

$$v(p) + \int_{C_o} v(p) \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q = 0 \qquad \text{for } p \in C_o \qquad (47)$$

where we have replaced $\alpha(p)u(p)$ by $v(p)$. Adopting the notation of Kleinman and Roach[9] we introduce the boundary integral operators on C_o

$$Kv := \int_{C_o} v(q) \frac{\partial \gamma(p,q)}{\partial n_p}\, ds_q \qquad \text{for } p \in C_o \qquad (48)$$

and

$$\bar{K}^*v := \int_{C_o} v(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in C_o \qquad (49)$$

so that Equation (47) may be written

$$(I+\bar{K}^*)\, v = 0 \qquad \text{for } p \in C_o \qquad (50)$$

In order to proceed we recall the jump properties of single and double layer potentials: if u is continuous on C_o (actually $L^1(C_o)$ suffices), then

$$\lim_{p\to C_o^\pm} \frac{\partial}{\partial n_p} \int_{C_o} u(q)\,\gamma(p,q)\, ds_q = \pm\, u(p) + \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_p} ds_q \qquad (51a)$$

$$\lim_{p\to C_o^\pm} \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = \mp\, u(p) + \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q \qquad (52a)$$

where $\lim_{p\to C_o^\pm}$ means that p approaches C_o from $D_\pm$. Since C_o may have corners, Equations (51a) and (52a) hold almost everywhere on C_o; see Lemma 4, Appendix B). The fact that γ is given by Equation (34) rather than merely being the fundamental solution $\gamma_o = -1/(2\pi|p-q|)$, does not alter these properties, since $\gamma - \gamma_o$ is regular in D_+, D_-, and C_o. That C_o is not closed is of no consequence provided C_o is piecewise Liapunov and satisfies a cone condition. If we further restrict C_o so that it is made up of segments lying on Liapunov surfaces of Holder index 1 ($|\hat{n}_q\cdot(p-q)|<A|p-q|^2$), then Lemma 6, Appendix B assures that if one of the derivatives

$$\frac{\partial}{\partial n_p^+} \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q \quad \text{or} \quad \frac{\partial}{\partial n_p^-} \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q$$

exists for $p \in C_o$, then the other does also and they are equal. We will henceforth assume that this slightly stronger smoothness requirement is satisfied, which still allows for corners and edges.

It will be useful to add the jump conditions for layers defined on C_w. If u is continuous on C_w (again $L^1(C_w)$ would suffice), then, since $\tilde{q} = q$ (see Equation (37)), the singularity at q is doubled in strength and

$$\gamma(p,q) = -\frac{1}{\pi|p-q|} + O(\ln|p-q|)$$

hence

$$\lim_{p \to C_w^-} \frac{\partial}{\partial n_p} \int_{C_w} u(q)\, \gamma(p,q)\, ds_q = -2u(p) + \int_{C_w} u(q) \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q \tag{51b}$$

$$\lim_{p \to C_w^-} \int_{C_w} u(q) \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q = 2u(p) + \int_{C_w} u(q) \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q \tag{52b}$$

where p approaches C_w from D_-, since we have not, as yet, defined $\gamma(p,q)$ outside the slab $-h \leq y_p(y_q) \leq 0$. The normal on C_w is taken to point in the positive y direction.

Another way to arrive at boundary integral equations for the floating body problem is to assume that a solution is given in the form of a single layer with unknown continuous density on C_o.

$$\phi(p) := \int_{C_o} u(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in D_+ \tag{53}$$

Then, with Lemma 1 in Appendix B we see that ϕ is continuous to the boundary (we may define $u = 0$ on $\partial D_+ \backslash C_o$). Also, ϕ satisfies: Laplace's equation, Equation (1b); the free surface condition, Equation (1c); the boundary condition on C_B, Equation (1d); and the radiation condition, Equation (1f). Moreover, with Lemma 9 of Appendix B we can show that $\phi \in H^1_{loc}(D_+)$. Finally, to satisfy the boundary condition on C_o we take the normal derivative, use the jump relation, Equation (51a), and arrive at the integral equation

$$u(p) + \int_{C_o} u(q) \frac{\partial \gamma(p,q)}{\partial n_p} ds_q = V(p) \qquad \text{for } p \text{ a.e. on } C_o \quad (54)$$

or, with Equation (48),

$$(I+K)u = V \qquad \text{for } p \in C_o \quad (55)$$

Following Kleinman and Roach,[9] we call those values of k for which $(I+K)u = 0$ has nontrivial continuous solutions irregular or characteristic values of $-K$ and values of k for which $(I+\bar{K}^*)u = 0$ has nontrivial solutions, we call irregular or characteristic values of $(-\bar{K}^*)$. Actually, Kleinman and Roach[9] (see also Angell and Kleinman[10] for clarification) treated solutions in $L^2(C_o)$ for smooth closed C_o in which case it is known that solutions of $(I+K)u = 0$, which lie in $L^2(C_o)$, also are continuous.

In the present case we will consider continuous or piecewise continuous solutions in order that normal derivatives of double layer distributions satisfy Lemma 6 (Appendix B). If K is compact, then these irregular values are the same. However, K is not compact in the space of continuous functions on C_o if the surface is not smooth. Nevertheless, some of the features of compact operators are retained. To see this we first define the adjoint floating body problem as the following interior problem: Find $\tilde{\phi} \in C^2(D_-)$ with interior boundary flow or in $H^1(D_-)$ such that

$$\nabla^2 \tilde{\phi} = 0 \qquad \text{for } p \in D_- \quad (56)$$

$$\frac{\partial \tilde{\phi}}{\partial y} - k\tilde{\phi} = 0 \qquad \text{for } p \in C_w \quad (57)$$

$$\tilde{\phi} = f \qquad \text{for } p \in C_o \tag{58}$$

where f is a given function.

Further, those values of k for which a nontrivial $\tilde{\phi}$ exists, such that Equations (56)-(58) hold with f = 0, will be called eigenvalues of the adjoint floating body problem.

We now prove:

Theorem 1 - The eigenvalues of the adjoint floating body problem are real.

Proof - Assume u is an eigenfunction. Then Green's theorem implies

$$\int_{C_o \cup C_w} \left(u \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial u}{\partial n} \right) ds = 0 \tag{59}$$

and, applying the boundary conditions, we get

$$\int_{C_w} (\bar{k}-k) \; |u|^2 \; ds = -2i \int_{C_w} \text{Im } k \; |u|^2 \; ds = 0 \tag{60}$$

hence, either u = 0 or Im k = 0. However, if u = 0 on C_w, then since u also vanishes on C_o, u is an eigenfunction of the interior Dirichlet problem and it is known that there are no nontrivial solutions of this problem for the Laplacian. Hence, Im k = 0 which proves the theorem. Next we prove:

Theorem 2: If k is an eigenvalue of the adjoint floating body problem, then k is a characteristic value of -K.

Proof: Assume k is such an eigenvalue so that there exists a nontrivial function v_- defined in D_- which satisfies Equations (56), (57), and (58) with f = 0. With Green's theorem, Equation (40), we have

$$\begin{aligned} 2v_- &= \int_{C_o \cup C_w} \left(v_- \frac{\partial \gamma}{\partial n_q} - \gamma \frac{\partial v_-}{\partial n_q} \right) ds_q && \text{for } p \in D_- \\ &= -\int_{C_o} \gamma(p,q) \frac{\partial v_-}{\partial n_q} \, ds_q && \text{for } p \in D_- \end{aligned} \tag{61}$$

where the boundary conditions on C_o, as well as the fact that both γ and v_- satisfy the free surface condition on C_w have been used. Now, taking the normal derivative and using the jump conditon of Equation (51a) we obtain

$$2\,\frac{\partial v_-}{\partial n} = \frac{\partial v_-}{\partial n} - \int_{C_o} \frac{\partial \gamma}{\partial n_p}\,\frac{\partial v_-}{\partial n_q}\,ds_q \qquad \text{for } p \in C_o \text{ a.e.} \tag{62}$$

or, with Equation (48),

$$(I+K)\,\frac{\partial v_-}{\partial n} = 0 \qquad \text{for } p \in C_o \tag{63}$$

Since both v_- and $\partial v_-/\partial n$ cannot both be zero on C_o because v_- is assumed a non-trivial solution of the Laplacian in D_-, it follows that k is a characteristic value of -K. Furthermore, we can also show:

Theorem 3: If Im $k \geq 0$ and k is a characteristic value of -K, then k is an eigenvalue of the adjoint floating body problem.

Proof: Assume $(I+K)\phi = 0$ has a nontrivial solution ϕ. Then define

$$v_+ := \int_{C_o} \phi(q)\,\gamma(p,q)\,ds_q \qquad \text{for } p \in D_+ \tag{64}$$

The jump condition, Equation (51a), then shows that

$$\frac{\partial v_+}{\partial n} = (I+K)\,\phi = 0 \tag{65}$$

and hence, the Uniqueness Theorem guarantees that $v_+ = 0$ for $p \in D_+$. Now let

$$v_- := \int_{C_o} \phi(q)\,\gamma(p,q)\,ds_q \qquad \text{for } p \in D_- \tag{66}$$

Since the single layer is continuous across C_o (Lemma 1, Appendix B), it follows that $v_- = 0$ on C_o and, because of the properties of the Green's function, either v_- is an eigenfunction of the adjoint floating body problem or $v_- = 0$ for $p \in D_-$. The latter possibility is ruled out since, if $v_- \equiv 0$ then $\partial v_-/\partial n^- = 0$ and, with Equations (51a) and (48), we would have

$$\frac{\partial v_-}{\partial n^-} = -\phi + K\phi = 0 \qquad \text{for } p \in C_o \qquad (67)$$

With Equation (65) this would imply $\phi = 0$ contrary to the assumption, thus proving the theorem. Theorems 2 and 3 establish the equivalence between eigenvalues of the adjoint floating body problem and characteristic values of $-K$. We can also obtain some relations, though not as complete, for characteristic values of $-\bar{K}^*$ as follows.

Theorem 4: If $\mathrm{Im}\ k \geq 0$ and k is a characteristic value of $-\bar{K}^*$, then k is an eigenvalue of the adjoint floating body problem, hence also a characteristic value of $(-K)$.

Proof: Let ϕ be a nontrivial solution of $(I+\bar{K}^*)\ \phi = 0$ and $p \in C_o$. Such a ϕ exists since k is assumed to be a characteristic value of $-\bar{K}^*$.

Define:

$$v_- := \int_{C_o} \phi(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_-$$

Then, using the jump condition, Equation (52a), we see that $v_- = (I+\bar{K}^*)\ \phi = 0$. Hence, because of the properties of γ, either v_- is an eigenfunction of the adjoint interior problem or $v_- \equiv 0$. The latter case is ruled out by the following argument. If $v_- = 0$ for $p \in D_-$, then $\partial v_-/\partial n^- = 0$ on C_o.

Now define

$$v_+ := \int_{C_o} \phi(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_+$$

and use the continuity of the normal derivative of the double layer (Lemma 6, Appendix B) to deduce that

$$\frac{\partial v_+}{\partial n} = 0 \qquad \text{for } p \in C_o$$

Then $v_+ = 0$ for $p \in D_+$ from the Uniqueness Theorem and in particular, from Equation (52a),

$$v_+ = (-I+\bar{K}^*)\ \phi = 0 \qquad \text{for } p \in C_o$$

Since it is also true that $(I+\bar{K}^*)\ \phi = 0$, it follows that $\phi = 0$, contrary to assumption, thus establishing that k is an eigenvalue of the adjoint floating body problem. Application of Theorem 2 completes the proof.

UNIQUELY SOLVABLE INTEGRAL EQUATIONS

We saw in the previous section that the integral equation formulation of the problem led to equations which were not uniquely solvable at irregular frequencies. Here we present two methods for modifying these equations to regain unique solvability. One method involves modifying the domain of the operator, whereas the second involves modifying the operator on the same domain.

Method 1: This method involves the use of Equation (38) not only on C_o, but in D_- as well. In scattering theory this is sometimes called the extended boundary condition method (see, e.g., Kupradze,[21] Schenck,[22] and Waterman[23] and has been used in slightly different form in the present context by Chang and Pien.[7] First we prove:

Theorem 5: If u is continuous and bounded on C_o and

$$\int_{C_o} u \frac{\partial\gamma(p,q)}{\partial n_q}\, ds_q = 0 \qquad \text{for } p \in D_- \cup C_w \quad (68)$$

then $u = 0$ on C_o.

Proof: Let

$$\phi_- = \int_{C_o} u \frac{\partial\gamma(p,q)}{\partial n_q}\, ds_q \qquad \text{for } p \in D_- \cup C_w$$

Then,

$$\phi_- = 0 \qquad \text{for } p \in D_- \cup C_w$$

and, using Equation (52a)

$$\lim_{p \to C_o^-} \phi_- = 0 = u(p) + \int_{C_o} u(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for p a.e. on } C_o \tag{69}$$

and

$$\frac{\partial \phi_-}{\partial n^-} = 0 \qquad \text{for p a.e. on } C_o$$

Now define

$$\phi_+ := \int_{C_o} u(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_+$$

With Lemma 6, Appendix B, on the continuity of normal derivatives of the double layer with continuous density, we have

$$\frac{\partial \phi_+}{\partial n^+} = 0 \qquad \text{for } p \in C_o$$

hence, ϕ_+ is a solution of the homogeneous floating body problem and by the Uniqueness Theorem, $\phi_+ = 0$ for $p \in D_+$. Again, with Equation (52a) we have

$$-u(p) + \int_{C_o} u(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_o \text{ a.e.} \tag{70}$$

Subtracting Equation (70) from Equation (69) establishes the theorem. This means that

$$\int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} \gamma(p,q) V(q) ds_q \quad \text{for } p \in D_- \cup C_w \tag{71}$$

has, at most, one solution, but the equation still presents problems since the domain and the range do not coincide and the range is all of $D_- \cup C_w$. However, we can improve things somewhat by applying Equation (38) only on the boundary of D_- and establish

Theorem 6: If u is continuous and bounded on C_o

$$u(p) + \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_o' \tag{72}$$

and

$$\int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_w' \tag{73}$$

then $u = 0$ for $p \in C_o \cup C_w$. We use C_o' and C_w' to denote the smooth parts of C_o and C_w.

Proof: Assume u is a solution of Equations (72) and (73). Define

$$\phi_-(p) := \int_{C_o} u(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_- \tag{74}$$

Then, since u is assumed continuous, with Equations (52a), (72), and (73) we see that $\phi_- = 0$ for $p \in C_o \cup C_w$. Hence, $\phi_- = 0$ for $p \in D_-$ since there are no eigenfunctions of the interior Dirichlet problem for Laplace's equation. Then the previous theorem shows that u = 0 on C_o. This theorem shows that even though k may be an irregular frequency, which means that Equation (72) may have nontrivial solutions, by adding the additional requirement of Equation (73), then uniqueness is assured at all frequencies. Thus, the inhomogeneous integral equations in the form

$$\phi(p) + \int_{C_o} \phi(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \quad \text{for } p \in C_o' \qquad (75)$$

and

$$\int_{C_o} \phi(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in C_w' \qquad (76)$$

have, at most, one continuous bounded solution, $\phi(p)$ for $p \in C_o$. We defer a proof of existence of a solution of these equations. These equations still suffer from the drawback that the unknown function ϕ is defined on C_o, but the equations must be satisfied on $C_o \cup C_w$. A more usual equation of the second kind with a kernel whose domain and range are the same may be obtained through the use of the following.

<u>Theorem 7</u>: If $\tilde{u}(p)$ is continuous and bounded on C_w, and

$$2\,\tilde{u}(p) + \int_{C_w} \tilde{u}(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_w \qquad (77)$$

then u(p) = 0. Here $\hat{n}_q$ points from C_w in the positive y-direction.

<u>Proof</u>: Assume $\tilde{u}$ satisfies Equation (77) and let

$$\tilde{\phi} := \int_{C_w} \tilde{u}(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_- \cup C_o \cup D_+$$

Then, because of the properties of γ, we see that $\tilde{\phi}$ satisfies Laplace's equation in the slab, $-h < y < 0$; the free surface condition, Equation (1c), on C_f; the homogeneous Neumann condition, Equation (1d), on C_B; the radiation condition, Equation (1f); and, because of Equation (77) and the jump condition, Equation (52b), $\tilde{\phi} = 0$ for $p \in C_w$.

Now apply the divergence theorem to $\bar{\tilde{\phi}}\ \nabla\tilde{\phi}$ over $D_{+} \cup C_o \cup D_{-}$ using the above relations obtaining

$$\int_{D_{+} \cup D_{-}} |\nabla\tilde{\phi}|^2\ d\tau = -\int_{C_f} \bar{\tilde{\phi}}\ \frac{\partial\tilde{\phi}}{\partial n}\ ds = k\int_{C_f} |\tilde{\phi}|^2\ ds \tag{78}$$

Now we follow precisely the part of the uniqueness proof following Equation (16), since $\tilde{\phi}$ defined above is of the form of Equation (17), and conclude that

$$\tilde{\phi} = 0 \qquad \text{for for } p \in D_{+} \cup C_o \cup D_{-}$$

Moreover, since γ satisfies the free surface condition on $y = 0$ we also have

$$\tilde{\phi} = 0 = k\int_{C_w} \tilde{u}(q)\ \gamma(p,q)\ ds_q \qquad \text{for } p \in D_{-} \tag{79}$$

or

$$\int_{C_w} \tilde{u}(q)\ \gamma(p,q)\ ds_q = 0 \qquad \text{for } p \in D_{-} \tag{80}$$

Now take the normal derivative form D_{-} using the jump condition, Equation (51b), obtaining

$$-2\ \tilde{u}(p) + \int_{C_w} \tilde{u}(q)\ \frac{\partial\gamma(p,q)}{\partial n_p}\ ds_q = 0 \qquad \text{for } p \in C_w \tag{81}$$

But with the representation of the Green's function, Equation (34), we see that, for p and $q \in C_w$ ($\Rightarrow y_p = y_q = 0$)

$$\frac{\partial\gamma}{\partial n_q} = \frac{\partial\gamma}{\partial n_p} := -i \sum_{n=0}^{\infty} \frac{k_n(k_n^2-k^2)}{hk_n^2-hk^2+k} H_o^{(1)} (k_n\widetilde{R}) \sinh k_n h \cosh k_n h \tag{82}$$

Thus, Equation (81) can be written

$$-2\,\widetilde{u}(p) + \int_{C_w} \widetilde{u}(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_w' \tag{83}$$

which, with Equation (77), implies that $\widetilde{u}(p) = 0$ for $p \in C_w$, thus establishing the theorem. This theorem shows that if Equation (77) holds, then necessarily

$$\int_{C_w} \widetilde{u}(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_o \tag{84}$$

Now combine Equations (75), (76), (83), and (84) and define

$$\Phi(p) := \begin{cases} \phi(p) \text{ for } p \in C_o \\ \widetilde{u}(p) \text{ for } p \in C_w \end{cases} \tag{85}$$

to obtain simply

$$\Phi(p) + \int_{C_o \cup C_w} \beta(q)\, \Phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in C_o' \cup C_w' \tag{86}$$

where $\beta(q) = \begin{cases} 1 \text{ for } q \in C_o \\ 1/2 \text{ for } q \in C_w \end{cases}$

Equation (86) is of the desired form and if it has a unique solution, then the restriction of Φ to C_o is the solution of Equations (75) and (76). It remains then to verify that Equation (86) has, at most, one solution which is the content of:

Theorem 8: If Φ_o is piecewise continuous on $C_o \cup C_w$ and

$$\Phi_o(p) + \int_{C_o \cup C_w} \beta(q)\, \Phi_o(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = 0 \quad \text{for } p \in C_o \cup C_w \tag{87}$$

then $\Phi_o = 0$

Proof: Assume Φ_o satisfies Equation (87) and define

$$v_- := \int_{C_o \cup C_w} \beta(q)\, \Phi_o(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_- \tag{88}$$

Then, using the jump conditions, Equations (52a) and (52b) and Equation (87), we find that $v_- = 0$ for $p \in \partial D_-$. Hence, $v_- = 0$ for $p \in D_-$ since the interior Dirichlet problem has no eigenfunctions. Thus,

$$\frac{\partial v_-}{\partial n} = 0 \qquad \text{for } p \in C_o \tag{89}$$

Now define

$$v_+ := \int_{C_o \cup C_w} \beta(q)\, \Phi_o(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q \qquad \text{for } p \in D_+ \tag{90}$$

Since v_+ is a double layer with piecewise continuous density, we may invoke Lemma 6, Appendix B, to see that

$$\frac{\partial v_+}{\partial n} = \frac{\partial v_-}{\partial n} = 0 \qquad \text{for } p \in C_o \tag{91}$$

But then v_+ is a solution of the homogeneous floating body problem and the Uniqueness Theorem then implies that

$$v_+(p) = 0 \qquad \text{for } p \in D_+ \quad (92)$$

Taking limiting values as $p \to C_o$ we have, with Equation (52a),

$$v_- = 0 \text{ for } p \in D_- \Rightarrow \Phi_o + \int_{C_o \cup C_w} \beta(q)\, \Phi_o(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_o \quad (93)$$

and

$$v_+ = 0 \text{ for } p \in D_+ \Rightarrow -\Phi_o + \int_{C_o \cup C_w} \beta(q)\, \Phi_o(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_o \quad (94)$$

from which we deduce that $\Phi_o = 0$ for $p \in C_o$. Using this result in Equation (87) we have

$$2\,\Phi_o(p) + \int_{C_w} \Phi_o(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 0 \qquad \text{for } p \in C_w \quad (95)$$

But then the unique solvability proven in Theorem 7 implies $\Phi_o(p) = 0$ for $p \in C_w$. Hence, $\Phi_o(p) = 0$ for $p \in C_o \cup C_w$, thus establishing the theorem.

To summarize the results of this section: <u>If $\phi(p)$ is a solution of the floating body problem Equations (1a)-(1f), then on C_o, $\phi(p) = \Phi(p)$ for $p \in C_o$ where Φ satisfies the equation</u>

$$\Phi(p) + \int_{C_o \cup C_w} \beta(q)\, \Phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in C_o' \cup C_w' \quad (96)$$

<u>and this equation has, at most, one solution.</u>

An alternate form of the equation may be derived when V(p) is the normal derivative of an interior potential; i.e.,

$$V(p) = -\frac{\partial\tilde{\phi}}{\partial n} \qquad \text{for } p \in C_o \tag{97}$$

For example, in heaving motion $V = -\hat{n} \cdot \hat{y}$, hence $\tilde{\phi} = y + C$. When Equation (97) holds, we may use the Green's identities, Equations (38) and (40), to write

$$\int_{C_o} \left\{ [\phi(q)+\tilde{\phi}(q)] \frac{\partial\gamma(p,q)}{\partial n_q} - \gamma(p,q) \frac{\partial}{\partial n_q} [\phi(q)+\tilde{\phi}(q)] \right\} ds_q$$

$$= 2\,\tilde{\phi}(p) - \alpha(p)\,[\tilde{\phi}(p)+\phi(p)] + \int_{C_w} \left[\gamma(p,q) \frac{\partial\tilde{\phi}}{\partial n_q} - \tilde{\phi}(q) \frac{\partial\gamma(p,q)}{\partial n_q}\right] ds_q \tag{98}$$

Now define

$$\phi_t(p) := \phi(p) + \tilde{\phi}(p) \tag{99}$$

and use the boundary conditions on C_o and C_f to obtain the representation

$$\alpha(p)\,\phi_t(p) + \int_{C_o} \phi_t(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 2\,\tilde{\phi}(p) + \int_{C_w} \gamma(p,q) \left[\frac{\partial\tilde{\phi}}{\partial n_q} - k\tilde{\phi}(q)\right] ds_q \tag{100}$$

The representation in Equation (100) is valid for all points in the slab $-h \leq y \leq 0$.

The major advantage of this form lies in the case of heaving motion where we may cause the integral over the waterplane to vanish by choosing the constant in $\tilde{\phi}$ appropriately, namely

$$\tilde{\phi} = y - \frac{1}{k} \tag{101}$$

Then, Equations (75) and (76) become

$$\phi_t(p) + \int_{C_o} \phi_t(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 2\left(y - \frac{1}{k}\right) \qquad \text{for } p \in C_o' \quad (102)$$

$$\int_{C_o} \phi_t(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = -\frac{2}{k} \qquad \text{for } p \in C_w' \quad (103)$$

whereas, in place of Equation (96) we have,

$$\Phi(p) + \int_{C_o \cup C_w} \beta(q)\, \Phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = 2\left(y - \frac{1}{k}\right) \quad (104)$$

$$\text{for } p \in C_o' \cup C_w'$$

where

$$\Phi(p) = \phi_t(p) \qquad \text{for } p \in C_o$$

Method 2: A second method of obtaining an integral equation with, at most, one solution which does not involve extending the domain of the integral operator is patterned after the method used in acoustic scattering problems by Burton and Miller,[8] Kleinman and Roach,[9] and Angell and Kleinman.[10] Again we start with the representation, Equation (38); evaluate it on C_o; take the normal derivative from D_+, using Equations (51a) and (52a); and obtain the pair of equations

$$\phi(p) + \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q)\, \gamma(p,q)\, ds_q \qquad \text{for } p \in C_o' \quad (105)$$

and

$$\frac{\partial}{\partial n_p} \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q = \int_{C_o} V(q) \frac{\partial\gamma(p,q)}{\partial n_p} ds_q - V(p) \quad (106)$$

$$\text{for } p \in C_o'$$

Since V is assumed to be in $L^1(C_o)$ then Lemma 5, Appendix B, ensures that the normal derivative of the right-hand side of Equation (105) exists in L^1 and since $\partial\phi/\partial n$ is also assumed to exist ($\partial\phi/\partial n$ is equal to V), then two of the three terms in Equation (38) have normal derivatives from D_+, hence the third term also must have a normal derivative. In addition, since ϕ is assumed continuous, Lemma 6, Appendix B, ensures that the normal derivative of the double layer,

$$\frac{\partial}{\partial n_p}\int_{C_o} \phi(q)\,\frac{\partial\gamma(p,q)}{\partial n_q}\,ds_q$$

exists almost everywhere on C_o and is the same whether we approach C_o from D_+ or D_-. Hence Equation (106) is obtained even though C_o may have corners. Denoting this normal derivative by D_n, i.e.,

$$D_n\phi \;:=\; \frac{\partial}{\partial n_p}\int_{C_o} \phi(q)\,\frac{\partial\gamma(p,q)}{\partial n_q}\,ds_q \tag{107}$$

we may rewrite Equations (105) and (106) as

$$(I+\bar{K}^*)\phi = \int_{C_o} V(q)\,\gamma(p,q)\,ds_q \qquad \text{for } p\in C_o' \tag{108}$$

$$D_n\phi = KV - V \qquad \text{for } p\in C_o' \tag{109}$$

Alternately, if V(p) is the normal derivative of an interior potential, Equation (97), then we may differentiate the representation of Equation (100) in the normal direction, obtaining

$$\frac{\partial}{\partial n_p}\int_{C_o} \phi_t(q)\,\frac{\partial\gamma(p,q)}{\partial n_q}\,ds_q = -2\,V(p) + \int_{C_w} \frac{\partial\gamma(p,q)}{\partial n_p}\left(\frac{\partial\tilde{\phi}}{\partial n_q} - k\tilde{\phi}\right) ds_q \tag{110}$$

$$\text{for } p\in C_o'$$

and the pair of boundary integral equations become

$$(I+\bar{K}^*)\ \phi_t = 2\ \tilde{\phi}(p) + \int_{C_w} \gamma(p,q) \left(\frac{\partial\tilde{\phi}}{\partial n_q} - k\tilde{\phi}(q)\right) ds_q \quad \text{for } p \in C_o' \quad (111)$$

$$D_n\ \phi_t = -2\ V(p) + \int_{C_w} \frac{\partial\gamma(p,q)}{\partial n_p} \left(\frac{\partial\tilde{\phi}}{\partial n_q} - k\tilde{\phi}(q)\right) ds_q \quad \text{for } p \in C_o' \quad (112)$$

As with Equation (100), we remark that this form is convenient in the case of heaving motion with $\tilde{\phi} = y - 1/k$ in which case the waterplane integrals vanish. It is the pair of equations, either Equations (108) and (109) or Equations (111) and (112), which will be shown to be uniquely solvable. First we observe:

Theorem 9: If k is not an eigenvalue of the adjoint floating body problem and ϕ is a piecewise continuous solution (in the closure of C_o) of

$$(I+\bar{K}^*)\phi = \int_{C_o} V(q)\ \gamma(p,q)\ ds_q \qquad \text{for } p \in C_o' \quad (113)$$

then

$$D_n\phi = KV - V \qquad \text{for } p \in C_o' \quad (114)$$

Proof: Let ϕ satisfy Equation (108). Then define

$$v_- := \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q - \int_{C_o} V(q)\ \gamma(p,q)\ ds_q \qquad \text{for } p \in D_- \quad (115)$$

and use the jump condition, Equation (52a), to see that $v_- = 0$ on C_o and, in fact, that either $v_- = 0$ for $p \in D_-$ or v_- is an eigenfunction of the adjoint floating body problem. But, since k is not an eigenvalue of this problem, $v_- = 0$ for $p \in D_-$ and hence has a normal derivative from D_-. The term

$$\int_{C_o} V(q)\ \gamma(p,q)\ ds_q$$

also has a normal derivative, Equation (51a), hence the normal derivative of the double layer exists and the theorem follows.

Thus, if k is not an eigenvalue of the adjoint floating body problem it is sufficient to solve Equation (108). A similar argument shows that under the same assumption on k, it suffices to solve Equation (111).

It remains to show that regardless of the value of k, the pair of Equations (108) and (109) or Equations (111) and (112) are uniquely solvable. In this section we concern ourselves only with uniqueness and prove

Theorem 10: If $\mathrm{Im}\, k \geq 0$, the only piecewise continuous solution of the pair

$$(I+\bar{K}^*)\phi = 0 \tag{116}$$

$$D_n\phi = 0 \tag{117}$$

is $\phi = 0$.

Proof: Assume ϕ satisfies Equations (116) and (117) and construct the function

$$v_+ := \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q}\, ds_q \qquad \text{for } p \in D_+ \tag{118}$$

Since Equation (117) is satisfied, it follows that

$$\frac{\partial v_+}{\partial n} = 0 \qquad \text{for } p \in C_o$$

Hence, $v_+ = 0$ for $p \in D_+$, otherwise it would be a nontrivial solution of the homogeneous floating body problem. Now take the limit as p approaches C_o from D_+ using the jump condition, Equation (52a), to obtain

$$\lim_{p \to C_o^+} v_+(p) = -\phi(p) + \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q}\, ds_q = 0 \tag{119}$$

This last equation may be written as

$$(-I+\bar{K}^*)\phi = 0 \qquad \text{for } p \in C_o \quad (120)$$

which, when used with Equation (116) implies that $\phi = 0$.

As a consequence of this result, it follows that Equations (108) and (109) (or Equations (111) and (112)) have, at most, one solution. Furthermore, Theorem 9 shows that when k is not an eigenvalue of the adjoint floating body problem and ϕ is a solution of Equation (108) it is also a solution of Equation (109). Similarly, if ϕ satisfies Equation (111) it also satisfies Equation (112).

The question of existence has not yet been settled, however, we conclude this section with one more uniqueness result which is advantageous from a numerical viewpoint in that it involves a single integral equation in contrast to a pair. This is patterned after a result for the Helmholtz equation by Burton and Miller.[8] The idea is to combine Equations (108) and (109) by multiplying one of the equations by a suitable constant (or function) η and adding it to the other, thus obtaining a single equation

$$(I+\bar{K}^*+\eta D_n)\phi = \int_{C_o} V(q)\ \gamma(p,q)\ ds_q + \eta\ (KV-V) \qquad \text{for } p \in C_o \quad (121)$$

Remarkably this equation has, at most, one solution if Im $\eta \neq 0$. We state this as

<u>Theorem 11</u>: If Im $\eta \neq 0$, Im $k = 0$, and ϕ is a piecewise continuous solution of

$$(I+\bar{K}^*+\eta D_n)\phi = 0 \qquad \text{for } p \in C_o \quad (122)$$

then $\phi = 0$.

<u>Proof</u>: Assume ϕ satisfies Equation (122) and define

$$v_- := \int_{C_o} \phi(q)\ \frac{\partial\gamma(p,q)}{\partial n_q}\ ds_q \qquad \text{for } p \in D_- \quad (123)$$

With the jump condition, Equation (52a), and Equation (123) we have

$$v_- = (I+\bar{K}^*)\phi = -\eta\, D_n\phi \qquad \text{for } p \in C_o \quad (124)$$

and

$$\frac{\partial v_-}{\partial n} = D_n\phi = -\frac{1}{\eta}\,(I+\bar{K}^*)\phi \qquad \text{for } p \in C_o \quad (125)$$

Since γ satisfies the free surface condition on C_w, v_- will also satisfy this condition. Applying Green's theorem on D_- we get

$$0 = \int_{C_o \cup C_w} \left(v_- \frac{\partial \bar{v}_-}{\partial n} - \bar{v}_- \frac{\partial v_-}{\partial n} \right) ds = \int_{C_o} (\bar{\eta}-\eta)\, |D_n\phi|^2\, ds \qquad (126)$$

where Equations (124) and (125) have been used. The integral over C_w vanishes since $\mathrm{Im}\, k = 0$, hence v_- and $\bar{v}_-$ satisfy the same free surface condition on C_w. Since $\bar{\eta} - \eta \neq 0$ on C_o, Equation (126) implies that

$$D_n\phi = 0 \qquad \text{for } p \in C_o \quad (127)$$

which, with Equation (122), also implies that

$$(I+\bar{K}^*)\phi = 0 \qquad \text{for } p \in C_o \quad (128)$$

Now define

$$v_+ := \int_{C_o} \phi(q)\, \frac{\partial \gamma(p,q)}{\partial n_q}\, ds_q \qquad \text{for } p \in D_+ \quad (129)$$

With Equation (127) and the continuity of the normal derivative of the double layer we find

$$\frac{\partial v_+}{\partial n} = D_n\phi = 0 \qquad (130)$$

Thus, $v_+ = 0$ for $p \varepsilon D_+$ otherwise it would violate the Uniqueness Theorem. Taking the limit as p approaches C_o from D_+, and using Equation (52a), we find

$$\lim_{p \to C_o^+} v_+ = (-I+\bar{K}^*)\phi = 0 \tag{131}$$

Equations (128) and (131) then imply that $\phi = 0$ as required and the proof is complete.

In summary, <u>if $\phi(p)$ is a solution of the floating body problem, Equations (1a)-(1f) then, on C_o, $\phi(p)$ satisfies the equation</u>

$$\phi(p) + \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q + \eta \frac{\partial}{\partial n_p} \int_{C_o} \phi(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q$$

$$= \int_{C_o} V(q) \left[\gamma(p,q) + \eta \frac{\partial\gamma(p,q)}{\partial n_p}\right] ds_q - \eta V(q) \tag{132}$$

<u>and this equation has, at most, one solution if Im $\eta \neq 0$.</u>

Alternately, if $V(p) = -\partial\tilde{\phi}/\partial n$ for $p \varepsilon C_o$ where $\tilde{\phi}$ is a solution of Laplace's equation in D_-, and $\phi_t(p) = \tilde{\phi}(p) + \phi(p)$, then ϕ_t satisfies the equation

$$\phi_t(p) + \int_{C_o} \phi_t(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q + \eta \frac{\partial}{\partial n_p} \int_{C_o} \phi_t(q) \frac{\partial\gamma(p,q)}{\partial n_q} ds_q$$

$$= 2\,\tilde{\phi}(p) + \int_{C_w} \left[\gamma(p,q) + \eta \frac{\partial\gamma(p,q)}{\partial n_p}\right]\left(\frac{\partial\tilde{\phi}}{\partial n_q} - k\tilde{\phi}\right) ds_q - 2\eta V(p) \tag{133}$$

for $p \varepsilon C_o$

and this equation has, at most, one solution. In the case of heaving motion, when $V(p) = -n_y = -\hat{n} \cdot \hat{y}$, this becomes

$$\phi_t(p) + \int_{C_o} \phi_t(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q + \eta \frac{\partial}{\partial n_p} \int_{C_o} \phi_t(q) \frac{\partial \gamma(p,q)}{\partial n_q} ds_q = 2\left(y - \frac{1}{k}\right) + 2\eta\, n_y \tag{134}$$

Having established the uniqueness results our task is not complete until we show that the integral Equations (96) or (132) actually do have a solution, however, this existence proof is not included here.

ACKNOWLEDGMENTS

This research was carried out while the author was a visitor at DTNSRDC. The congenial and stimulating environment was much appreciated. Also gratefully acknowledged are the illuminating discussions of this problem with Drs. K. June Bai, Ming S. Chang, and Choung M. Lee which contributed greatly to the author's understanding.

APPENDIX A
A TRANSCENDENTAL EQUATION

We list here some properties of the roots of the equation defining the wave frequencies

$$k_n \sinh k_n h = k \cosh k_n h \tag{A.1}$$

Property 1: If k_n is a root, then $-k_n$ is a root.

Property 2: If Im $k = 0$ and Re $k \neq 0$, then there exist exactly two distinct real roots and an infinite number of purely imaginary roots. This was observed by Fritz John.

Property 3: If Im $k \neq 0$, then Re $k_n \neq 0$. This may be seen by assuming that the statement is not true i.e., Re $k_n = 0$ or that $k_n = ix_n$, in which case, Equation (A.1) becomes

$$-x_n \sin x_n h = k \cos x_n h \tag{A.2}$$

Since the left-hand side is real and Im $k \neq 0$, we find $\cos x_n h = 0$ in which case, from Equation (A.2), $x_n \sin x_n h = 0$ and these two equations cannot be true simultaneously.

Properties 1 through 3, considered together, establish

Property 4: If $k \neq 0$, there exists at least one root of Equation (A.1) with a positive real part.

Property 5: If k is bounded, then Re k_n is bounded. This may be seen as follows. Assume k_n is a root of Equation (A.1), such that $|k_n| \geq 2|k|$. Then, solving Equation (A.1) for $e^{k_n h}$ we see that

$$e^{2k_n h} = \frac{k_n+k}{k_n-k} \text{ and } e^{-2k_n h} = \frac{k_n-k}{k_n+k} \tag{A.3}$$

from which it follows that

$$e^{2h|\mathrm{Re}k_n|} \leq \frac{|k_n|+|k|}{|k_n|-|k|} \tag{A.4}$$

Since the right-hand side is monotonically decreasing in $|k_n|$ for $|k_n| \geq 2|k|$ we have

$$e^{2h|\mathrm{Re}k_n|} \leq \frac{2|k|+|k|}{2|k|-|k|} = 3 \tag{A.5}$$

or

$$|\mathrm{Re}k_n| \leq \frac{1}{2h} \ln 3 \tag{A.6}$$

Recalling that this was found under the assumption that $|k_n| \geq 2|k|$ we see that either Equation (A.6) holds or

$$|k_n| < 2|k|$$

which implies

$$|\mathrm{Re}k_n| < 2|k|$$

thus,

$$|\mathrm{Re}k_n| \leq \max \left\{2|k|, \frac{1}{2h} \ln 3\right\} \tag{A.7}$$

which establishes Property 5. Not only is $\mathrm{Re}k_n$ bounded, but, as can be seen from Equation (A.4),

<u>Property 6:</u> $\lim\limits_{|k_n| \to \infty} \mathrm{Re}k_n = 0$

This helps establish

Property 7: There exists a root of Equation (A.1) with largest positive real part. We designate the root as k_o.

This follows since any bounded sequence of roots is either finite, hence has a member with largest real part, or converges to a bounded root and again there is one with largest real part. The bound may be chosen, with Property 6, so that the roots with magnitude larger than the bound have real parts smaller than $\mathrm{Re}k_o$. Finally we add

Property 8: If $\mathrm{Re}k_n > 0$ and $\mathrm{Im}\, k \geq 0$, then $\mathrm{Im}\, k_n \geq 0$. This can be seen by letting $k_n = x_n + iy_n$ with $x_n > 0$ and rewriting Equation (A.1) as

$$k = k_n \frac{\sinh k_n h}{\cosh k_n h} = \frac{x_n \sinh 2x_n h - y_n \sin 2y_n h + i(y_n \sinh 2x_n h + x_n \sin 2y_n h)}{2|\cosh (x_n + iy_n)h|^2} \tag{A.8}$$

But, $\mathrm{Im}\, k \geq 0$ implies that $y_n \sinh 2x_n h + x_n \sin 2y_n h \geq 0$. Recall that $x_n > 0$ and assume that $y_n < 0$. Dividing by $2h\, x_n\, y_n$ yields

$$\frac{\sinh 2x_n h}{2x_n h} + \frac{\sin 2y_n h}{2y_n h} \leq 0$$

hence,

$$\frac{\sinh 2x_n h}{2x_n h} \leq - \frac{\sin 2y_n h}{2y_n h} \leq \left| \frac{\sin 2y_n h}{2y_n h} \right| \leq 1$$

which is impossible with $x_n \neq 0$. Hence, $y_n > 0$ establishing the desired result.

APPENDIX B

SOME PROPERTIES OF SINGLE AND DOUBLE LAYER POTENTIALS ON NONSMOOTH BOUNDARIES

If C is a simply connected piecewise Liapunov surface, that is, $C = \bigcup_{i=1}^{n} C_i \cup \bigcup_{i=1}^{n} \partial C_i$, where each C_i lies on a Liapunov surface σ_i (e.g., Günter[13]), then many of the familiar properties of single and double layer potentials on smooth boundaries are still valid as shown by Wendland.[6] We list some of these useful properties here with some indication of how they are obtained. First we define the symbols:

$$S_o\mu := \int_C \mu(q)\, \gamma_o(p,q)\, ds_q \qquad \text{for } p \in \mathbb{R}^3 \tag{B.1}$$

$$D_o\mu := \int_C \mu(q)\, \frac{\partial \gamma_o(p,q)}{\partial n_q}\, ds_q \qquad \text{for } p \in \mathbb{R}^3 \tag{B.2}$$

$$K_o\mu := \int_C \mu(q)\, \frac{\partial \gamma_o(p,q)}{\partial n_p}\, ds_q \qquad \text{for } p \in C_i, \text{ for some } i \tag{B.3}$$

where

$$\gamma_o(p,q) := -\frac{1}{2\pi|p-q|} \tag{B.4}$$

We understand C_i to denote an open set on C so that corner and edge points are not included in C_i, but are confined to ∂C_i. Note that

$$D_o\mu = K_o^*\mu \qquad \text{for } p \in C$$

where

$$K_o^*\mu := \int_C \mu(q)\, \frac{\partial \gamma_o(p,q)}{\partial n_q}\, ds_q \qquad \text{for } p \in C \tag{B.5}$$

While μ is defined only on C, we extend by 0 onto σ_i, that is, define

$$\begin{aligned} \mu_i(p) &:= \mu(p) \qquad \text{for } p \in C_{i\cup}\partial C_i \\ &:= 0 \qquad \text{for } p \in \sigma_i \setminus (C_{i\cup}\partial C_i) \end{aligned} \tag{B.6}$$

Also, if α_i is the Holder index of the surface σ_i, let $\alpha = \min_i \{\alpha_i\}$ so that α may serve as a common index for all σ_i.

<u>Lemma 1:</u> $\mu \in L^\infty(C) \Rightarrow S_o\mu \in C^{o,\alpha}(R^3) \Rightarrow S_o\mu \in L^p(C)$ for $p \geq 1$.

<u>Proof:</u> Angell and Kleinman[10] show that the result holds if C is Liapunov with index α. But,

$$\begin{aligned} S_o\mu &= \sum_{i=1}^{n} \int_{C_i} \mu(q)\, \gamma_o(p,q)\, ds_q = \sum_{i=1}^{n} \int_{\sigma_i} \mu_i(q)\, \gamma_o(p,q)\, ds_q \\ &= \sum_{i=1}^{n} S_{oi}\, \mu_i \end{aligned} \tag{B.7}$$

where

$$S_{oi}\, \mu_i = \int_{\sigma_i} \mu_i(q)\, \gamma_o(p,q)\, ds_q \tag{B.8}$$

But

$$\mu \in L^\infty(C) \Rightarrow \mu_i \in L^\infty(\sigma_i)$$

hence

$$S_{oi}\, \mu_i \in C^{o,\alpha}(R^3)$$

for each i as does a finite linear combination, thus establishing the result.

Lemma 2: $\mu \in L^\infty(C) \Rightarrow K_o^*\mu \in L^p(C)$ for $p \geq 1$.

Proof: As before, we rewrite the operator as a sum of operators

$$K_o^*\mu = \sum_{i=1}^{n} \int_{C_i} \mu_i(q) \frac{\partial \gamma_o(p,q)}{\partial n_q} ds_q \tag{B.9}$$

Now, for any $p \in \mathbb{R}^3$

$$\left| \int_{C_i} \mu_i(q) \frac{\partial \gamma_o(p,q)}{\partial n_q} ds_q \right| \leq \frac{1}{2\pi} \, ||\mu||_{L^\infty(C)} \int_{C_i} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3} ds_q \tag{B.10}$$

Rewrite the integral on the right as

$$\int_{C_i} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3} ds_q = \int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3} ds_q + \int_{C_i \setminus (C_i \cap B_\delta(p))} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3} ds_q \tag{B.11}$$

where $B_\delta(p)$ is a ball of radius δ with the center at p, and we choose δ to be less than half the Liapunov radius of the Liapunov surface σ_i of which C_i is a part. Since

$$|p-q| > \delta \qquad \text{for } q \in C_i \setminus (C_i \cap B_\delta(p)) \tag{B.12}$$

$$\int_{C_i \setminus (C_i \cap B_\delta(p))} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3} ds_q \leq \int_{C_i \setminus (C_i \cap B_\delta(p))} \frac{ds_q}{|p-q|^2} \leq \frac{M(C_i)}{\delta^2} \tag{B.13}$$

where $M(C_i)$ is the surface area of C_i. To evaluate the first integral on the right of Equation (B.11) we consider two cases. If $p \in C_i \subset \sigma_i$, there exist constants A and α such that

$$|\hat{n}_q \cdot (p-q)| \le A\, |p-q|^{1+\alpha} \tag{B.14}$$

hence

$$\int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3}\, ds_q \le A \int_{C_i \cap B_\delta(p)} \frac{ds_q}{|p-q|^{2-\alpha}} \le 2A \int_0^\delta d\rho \int_0^{2\pi} d\phi\, \frac{\rho}{\rho^{2-\alpha}} \le A' \delta^\alpha \tag{B.15}$$

for some A' where we used the standard method of projection onto the plane tangent to C_i at p. Here A' is independent of p and δ. If $p \notin C_i$ then, denoting by $\widetilde{p}$ the normal projection of p onto σ_i, we have

$$\int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3}\, ds_q = \int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (p-\widetilde{p}+\widetilde{p}-q)|}{|p-q|^3}\, ds_q$$

$$\le |p-\widetilde{p}| \int_{C_i \cap B_\delta(p)} \frac{ds_q}{|p-q|^3} + \int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (\widetilde{p}-q)|}{|p-q|^3}\, ds_q \tag{B.16}$$

Note that $|p-\widetilde{p}| < \delta$ even if $\widetilde{p} \notin C_i$ and that $B_\delta(p) \subset B_{2\delta}(\widetilde{p})$. Using this fact and Equation (B.14) which holds with $\widetilde{p}$ replacing p,

$$\int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3}\, ds_q \le |p-\widetilde{p}| \int_{C_i \cap B_{2\delta}(\widetilde{p})} \frac{ds_q}{|p-q|^3} + A \int_{C_i \cap B_{2\delta}(\widetilde{p})} \frac{|\widetilde{p}-q|^{1+\alpha}}{|p-q|^3}\, ds_q \tag{B.17}$$

Now we use the inequality (e.g., Günter[13])

$$\frac{1}{|p-q|} \le \frac{2}{(|p-\widetilde{p}|^2+|\widetilde{p}-q|^2)^{1/2}} \tag{B.18}$$

to show that

$$\frac{|\tilde{p}-q|^{1+\alpha}}{|p-q|^3} \le \frac{8|\tilde{p}-q|^{1+\alpha}}{(|\tilde{p}-p|^2+|\tilde{p}-q|^2)^{[3-(1+\alpha)]/2}\,(|\tilde{p}-p|^2+|\tilde{p}-q|^2)^{(1+\alpha)/2}} \quad \frac{8}{|\tilde{p}-q|^{2-\alpha}} \tag{B.19}$$

Employing Equations (B.18) and (B.19) in Equation (B.17), we obtain

$$\int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (\tilde{p}-q)|}{|p-q|^3}\, ds_q \le 2|p-\tilde{p}| \int_{C_i \cap B_{2\delta}(\tilde{p})} \frac{ds_q}{(|p-\tilde{p}|^2+|\tilde{p}-q|^2)^{3/2}}$$

$$+ 8A \int_{C_i \cap B_{2\delta}(\tilde{p})} \frac{ds_q}{|\tilde{p}-q|^{2-\alpha}}$$

$$\le 2|p-\tilde{p}| \int_0^{2\delta} d\rho \int_0^{2\pi} d\phi\, \frac{\rho}{(|p-\tilde{p}|^2+\rho^2)^{3/2}} + 8A \int_0^{2\delta} d\rho \int_0^{2\pi} d\phi\, \frac{1}{\rho^{1-\alpha}}$$

$$\le -\left.\frac{4\pi|p-\tilde{p}|}{(|p-\tilde{p}|^2+\rho^2)^{1/2}}\right|_{\rho=0}^{2\delta} + \frac{16A}{\alpha}(2\delta)^\alpha \tag{B.20}$$

from which we infer that there are constant A_i and B_i independent of p and δ such that

$$\int_{C_i \cap B_\delta(p)} \frac{|\hat{n}_q \cdot (p-q)|}{|p-q|^3}\, ds_q \le A_i + B_i\, \delta^\alpha \tag{B.21}$$

Although Equation (B.21) was derived assuming that $p \notin C_i$, we see, with Equation (B.15), that Equation (B.21) remains valid even if $p \in C_i$. Employing the estimates of Equations (B.13) and (B.21) in Equation (B.10), we have

$$\left|\int_{C_i} \mu(q)\,\frac{\partial\gamma_o(p,q)}{\partial n_q}\,ds_q\right| \leq \frac{1}{2\pi}\,||\mu||_{L^\infty(C)}\left[\frac{M(C_i)}{\delta^2} + A_i + B_i\,\delta^\alpha\right] \tag{B.22}$$

Setting

$$M = \frac{1}{2\pi}\max_i \left\{\frac{M(C_i)}{\delta^2} + A_i + B_i\,\delta^\alpha\right\} \tag{B.23}$$

we have finally, from Equation (B.9), that

$$|K_o^*\mu| \leq \sum_{i=1}^{n} \left|\int_{C_i} \mu(q)\,\frac{\partial\gamma_o(p,q)}{\partial n_q}\,ds_q\right| \leq nM\,||\mu||_{L^\infty(C)} \tag{B.24}$$

Thus, $\mu \in L^\infty(C) \Rightarrow K_o^*\mu$ is bounded, hence, $K_o^*\mu \in L^p(C)$ for $p \geq 1$, which concludes the proof of Lemma 2. By Fubini's Theorem we then obtain

<u>Lemma 3:</u>* $\mu \in L^p(C) \Rightarrow K_o\mu \in L^1(C)$ for $p \geq 1$.

Next we have

<u>Lemma 4:</u> If $\mu \in L^1(C)$, then

$$\lim_{p\to C^{\mp}} \frac{\partial}{\partial n_p} S_o\mu = \mp\,\mu(p) + K_o\mu \qquad \text{for } p \text{ a.e. on } C \tag{B.25}$$

and

$$\lim_{p\to C^{\mp}} D_o\mu = \pm\,\mu(p) + K_o^*\mu \qquad \text{for } p \text{ a.e. on } C \tag{B.26}$$

---

*We remark that $K_o\mu$ need not be bounded (e.g., Fichera and Sneider-Ludovici,[24] Leis,[25] and Craggs, Mangler and Zamir[26]).

Here the $\pm$ denotes limits from either side of C. We choose + to denote the portion of $\mathbb{R}^3$ into which the normal from C points, call it D_+; then $D_- = \mathbb{R}^3 \setminus D_+ \cup C$

Proof: Since the results are to hold almost everywhere, corner and edge points of C may be excluded and if p is a smooth point of C_i, then

$$\lim_{p \to C^{\mp}} \frac{\partial}{\partial n_p} S_o \mu = \lim_{p \to C^{\mp}} \sum_{j=1}^{n} \frac{\partial}{\partial n_p} \int_{C_j} \mu(q)\, \gamma(p,q)\, ds_q$$

$$= \sum_{j=1}^{n}{}' \int_{C_j} \mu(q) \frac{\partial \gamma_o(p,q)}{\partial n_p} ds_q + \lim_{p \to C^{\mp}} \frac{\partial}{\partial n_p} \int_{C_i} \mu(q)\, \gamma_o(p,q)\, ds_q$$

$$= \sum_{j=1}^{n}{}' \int_{C_j} \mu(q) \frac{\partial \gamma_o(p,q)}{\partial n_p} ds_q + \lim_{p \to C^{\mp}} \frac{\partial}{\partial n_p} \int_{\sigma_i} \mu_i(q)\, \gamma_o(p,q)\, ds_q \tag{B.27}$$

where Σ' means that the i^{th} term is omitted.

Since σ_i is Liapunov, the validity of the jump relations (Günter[13]) shows that

$$\lim_{p \to C^{\mp}} \frac{\partial}{\partial n_p} S_o \mu = \sum_{j=1}^{n}{}' \int_{C_j} \mu(q) \frac{\partial \gamma_o(p,q)}{\partial n_p} ds_q \mp \mu(p) + \int_{\sigma_i} \mu_i(q) \frac{\partial}{\partial n_p} \gamma_o(p,q)\, ds_q$$

$$= \sum_{j=1}^{n}{}' \int_{C_j} \mu(q) \frac{\partial \gamma_o(p,q)}{\partial n_p} ds_q \mp \mu(p) + \int_{C_i} \mu(q) \frac{\partial}{\partial n_p} \gamma_o(p,q)\, ds_q$$

$$= \int_{C} \mu(q) \frac{\partial \gamma_o(p,q)}{\partial n_p} ds_q \mp \mu(p) \tag{B.28}$$

A similar decomposition establishes the jump condition for the double layer.

As a consequence of Lemmas 2, 3, and 4, we have

__Lemma 5:__ If $\mu \varepsilon L^{\infty}(C)$, then

$$\lim_{p \to C^{\mp}} \frac{\partial}{\partial n_p} S_o \mu \;\varepsilon\; L^1(C) \tag{B.29}$$

and

$$\lim_{p \to C^{\mp}} D_o \mu \;\varepsilon\; L^P(C) \qquad \text{for } p \geq 1 \tag{B.30}$$

This follows immediately since $\mu \varepsilon L^{\infty}(C)$ implies that Lemma 4 holds and also Lemmas 2 and 3, hence μ and $K_o^* \mu$ are in $L^P(C)$ and $K_o \mu \varepsilon L^1(C)$. Also of interest is the normal derivative of the double layer for which we may state

__Lemma 6:__ If $\mu \varepsilon C^o(C_i)$ for every i, σ_i is Liapunov with index 1, (which means $|\hat{n}_q \cdot (p-q)| < A|p-q|^2$), and if one of the derivatives $\partial D_o \mu / \partial n^+$ or $\partial D_o \mu / \partial n^-$ exists at a point $p \varepsilon C$, then the other exists and they are equal.

__Proof:__ We obviously exclude corner and edge points since there is no normal at such points. If p is an interior point of C_i, then the proof in the smooth case (Günter p 297 et eq.[13]) may be repeated without change since the crucial step involves treatment of a small patch around p which, in this case as in the smooth case, lies on an appropriately smooth Liapunov surface. The same argument shows that the result holds for piecewise continuous densities even if the discontinuities occur at smooth points, provided all points of discontinuity are excluded.

Finally, we list some properties of single and double layers considered as operators mapping functions defined on C, not to functions on C, but to functions defined on the interior of C which we denote as D_-. From Lemma 1 we immediately have

__Lemma 7:__ $\mu \varepsilon L^{\infty}(C) \Rightarrow S_o \mu \varepsilon L^P(D_-)$ for $p \geq 1$.

Also, since $D_o \mu \varepsilon C^{\infty}(D_-)$ and $\lim_{p \to C^-} D_o \mu \varepsilon L^{\infty}(C)$

(Lemmas 2 and 4), for $\mu \varepsilon L^{\infty}(C)$, we have

Lemma 8: $\mu \in L^{\infty}(C) \Rightarrow D_o\mu \in L^p(D_-)$ for $p \geq 1$.

More difficult to establish are properties of derivatives of potentials. However, we do have

Lemma 9: $\mu \in L^{\infty}(C) \Rightarrow \nabla S_o\mu \in L^2(D_-)$

Proof: For $p \in D_-$, we have

$$|\nabla S_o\mu| \leq \frac{1}{2\pi}\int_C |\mu(q)| \left|\frac{q-p}{|p-q|^3}\right| ds_q \leq \frac{1}{2\pi} ||\mu||_{L^{\infty}(C)} \int_C \frac{ds_q}{|p-q|^2} \tag{B.31}$$

therefore,

$$\int_{D_-} |\nabla S_o\mu|^2 \, d\tau_p \leq \frac{1}{(2\pi)^2} ||\mu||^2_{L^{\infty}(C)} \int_{D_-}\int_C \frac{ds_q}{|p-q|^2}\int_C \frac{ds_{q_1}}{|p-q_1|^2} d\tau_p$$

$$\leq \frac{1}{(2\pi)^2} ||\mu||^2_{L^{\infty}(C)} \int_C\int_C\int_{D_-} \frac{d\tau_p}{|p-q|^2|p-q_1|^2} ds_q \, ds_{q_1} \tag{B.32}$$

But

$$\int_{D_-} \frac{d\tau_p}{|p-q|^2|p-q_1|^2} \leq \int_{B_d(0)} \frac{d\tau_p}{|p|^2|p+q-q_1|^2} \tag{B.33}$$

where d is twice the diameter of D_-. Explicitly

$$\int_{D_-} \frac{d\tau_p}{|p-q|^2|p-q_1|^2} \leq \int_0^d dr\int_0^{\pi} d\theta\int_0^{2\pi} d\phi \frac{\sin\theta}{r^2+|q-q_1|^2-2p\cdot(q_1-q)}$$

$$\leq \int_0^{|q-q_1|} dr\int_0^{\pi} d\theta\int_0^{2\pi} d\phi \sin\theta \left[\sum_{n=0}^{\infty} \frac{r^n}{|q-q_1|^{n+1}} P_n(\cos\gamma)\right]^2 +$$

(B.34)
(cont.)

$$+ \int_{|q-q_1|}^{d} dr \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \, \sin\theta \left[\sum_{n=0}^{\infty} \frac{|q-q_1|^n}{r^{n+1}} P_n(\cos\gamma) \right]^2 \tag{B.34}$$

where we have employed the generating function for Lengendre polynomials and

$$\cos\gamma = \frac{p\cdot(q_1-q)}{|p\cdot(q_1-q)|} \tag{B.35}$$

Using the orthgonality of Legendre functions leads to

$$\int_{D_-} \frac{d\tau_p}{|p-q|^2|p-q_1|^2} \leq \int_0^{|q-q_1|} dr \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \frac{r^{2n}}{|q-q_1|^{2n+2}}$$

$$+ \int_{|q-q_1|}^{d} dr \sum_{n=0}^{\infty} \frac{4\pi}{2n+1} \frac{|q-q_1|^{2n}}{r^{2n+2}}$$

$$\leq 4\pi \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left\{ \frac{r^{2n+1}}{|q-q_1|^{2n+2}} \Bigg|_{r=0}^{|q-q_1|} - \frac{|q-q_1|^{2n}}{r^{2n+1}} \Bigg|_{r=|q-q_1|}^{d} \right\}$$

$$\leq \frac{8\pi}{|q-q_1|} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{12\pi}{|q-q_1|} \tag{B.36}$$

Substituting this estimate in Equation (B.32) we obtain

$$\int_{D_-} |\nabla S_o\mu|^2 \, d\tau_p \le \frac{3}{\pi} ||\mu||^2_{L^\infty(C)} \int_C \int_C \frac{ds_q \, ds_{q_1}}{|q-q_1|}$$

$$\le \frac{3}{\pi} ||\mu||^2_{L^\infty(C)} \int_C S_o 1 \, ds_q \tag{B.37}$$

But Lemma 1 ensures us that the single layer with density 1 is bounded as is its integral over C, hence

$$\int_{D_-} |\nabla S_o\mu|^2 \, d\tau_p \le \frac{3}{\pi} ||\mu||^2_{L^\infty(C)} \; ||S_o 1||_{L^1(C)} \tag{B.38}$$

which completes the proof of the Lemma.

It should be noted that no claim is made about the sharpness of the results of this Appendix. Indeed more complete and precise estimates of the mapping properties of single and double layers may become available (see, for example, Fabes, Jodeit, and Riviere[27]).

REFERENCES

1. John, F., "On the Motion of Floating Bodies I," Comm. Pure Appl. Math., II, pp. 13-52 (1949).

2. John, F., "On the Motion of Floating Bodies II," Comm. Pure Appl. Math., III, pp. 45-101 (1950).

3. Grisvard, P., "Boundary Value Problems in Non-Smooth Domains," University of Maryland Math. Department Lecture Notes, No. 19 (1980).

4. Burago, Y.D., V.G. Maz'ja and V.D. Sapozhnikova, "On the Theory of Simple and Double Layer Potentials for Domains with Irregular Boundaries," in Problems in Mathematical Analysis, Vol. I, Boundary Value Problems and Integral Equations, V.I. Smirnov, ed., Consultants Bureau, New York, pp. 1-30 (1968).

5. Král, J., "The Fredholm Method in Potential Theory," Trans. Amer. Math. Soc., 125, pp. 511-547 (1966).

6. Wendland, W., "Die Behandlung von Randwertaufgaben imR_3 mit Hilfe von Einfach und Doppelschicht Potentialen," Numerical Math., 11, pp. 380-404 (1968).

7. Chang, M.S. and P.C. Pien, "Hydrodynamic Forces on a Body Moving Beneath a Free Surface," Proceedings First International Conference on Numerical Ship Hydrodynamics, David Taylor Naval Ship Research and Development Center (1975).

8. Burton, A.J. and G.F. Miller, "The Application of Integral Equation Methods to the Numerical Solution of Some Exterior Boundary Value Problems," Proc. Royal Soc. Ser. A, 323, pp. 201-210 (1971).

9. Kleinman, R.E. and G.F. Roach, "Boundary Integral Equations for the Three-Dimensional Helmholtz Equation," SIAM Review, 16, pp. 214-236 (1974).

10. Angell, T.S. and R.E. Kleinman, "Boundary Integral Equations for the Helmholtz Equation," The Third Boundary Value Problem, Math. Meth. in the Appl. Sci., 4, pp. 164-193 (1982).

11. Ursell, F., "Irregular Frequencies and the Motion of Floating Bodies," Journal of Fluid Mechanics, 105, pp. 143-156 (1981).

12. Jones, D.S., "Integral Equations for the Exterior Acoustic Problem," Quarter J. Mech. Appl. Math., 27, pp. 129-142 (1974).

13. Günter, N.M., "Potential Theory and Its Applications to Basic Problems of Mathematical Physics," Ungar, New York (1967).

14. Kleinman, R.E. and W. Wendland, "On Neumann's Method for the Exterior Neumann Problem for the Helmholtz Equation," J. Math. Analysis and Applications, 57, pp. 170-202 (1977).

15. Lighthill, M.J., "On Waves Generated in Dispersive Systems by Travelling Forcing Effects, with Applications to the Dynamics of Rotating Fluids," J. Fluid Mech., 27, pp. 225-252 (1967).

16. Noblesse, F., "On the Theory of Flow of Regular Water Waves About a Body," MIT Department of Ocean Engineering Report 80-2 (1980).

17. Lenoir, M. and D. Martin, "An Application of the Principle of Limiting Absorption to the Motions of Floating Bodies," J. Math. Analysis and Applications, 79, pp. 370-383 (1981).

18. Ursell, F.J., "Note on a paper by M. Lenoir and D. Martin," Proceedings of the Symposium on Numerical Ship Hydrodynamics, Paris (1981) (in publication).

19. Burago, Y.D. and V.G. Maz'ja, "Potential Theory and Function Theory for Irregular Regions," Seminars in Mathematics, V.A. Steklov Math. Institute, Leningrad, 3, 1967, Consultants Bureau, New York (1969).

20. Ahner, J.F. and R.E. Kleinman, "The Exterior Neumann Problem for the Helmholtz Equation," Arch. Rational Mech. Anal., 52, pp. 26-43 (1973).

21. Kupradze, V.D., "On the Approximate Solution of Problems in Mathematical Physics," Russian Math. Surveys, 22, pp. 58-108 (1967).

22. Schenck, H.A., "Improved Integral Formulation for Acoustic Radiation Problems," J. Acoust. Soc. Amer., 44, pp. 41-58 (1968).

23. Waterman, P.C., "New Formulation of Acoustic Scattering," J. Acoust. Soc. Amer., 45, pp. 1417-1429 (1969).

24. Fichera, A. and M. Sneider-Ludovici, "Distribution de la Charge Électrique dans le Voisinage des Sommets et des Arêts dûn Cube," Rendiconti di Matem (1), 8, pp. 201-224 (1975).

25. Leis, R., "On the Dirichlet Problem for the Reduced Wave Equation," New York Univ. Inst. of Math. Sci. Res. Report BR-30 (1959).

26. Craggs, J.W., K.W. Mangler and M. Zamir, "Some Remarks on the Behavior of Surface Source Distributions Near the Edge of a Body," R.A.E. Farnborough Technical Report 71085 (1971).

27. Fabes, E.B., M. Jodeit and N.M. Rivière, "Potential Techniques for Boundary Value Problems on C^1 Domains," Acta Math., 141, pp. 165-186 (1978).

INITIAL DISTRIBUTION

Copies	
1	US Army Waterways Experiment Station Res Center Lib
1	CHONR/Code 438
1	ONR Boston
1	ONR Chicago
1	ONR Pasadena
1	USNA/P. Van Mater
1	NRL/Lib
1	NAVPGSCOL
1	NROTC & NAVADMINU
1	NAVWARCOL
4	NAVSEA 1 SEA 31B 1 SEA 321 2 SEA 3213
1	NAVFACENGCOM
1	NAVOCEANO/Lib
1	NWC
1	NOSC
1	NSWC, White Oak/Lib
1	NSWC, Dahlgren/Lib
1	NUSC NPT
1	NUSC NLONLAB
1	CEL/Code L31
1	NADC

Copies	
1	NAVSHIPYD BREM/Lib
1	NAVSHIPYD CHASN/Lib
1	NAVSHIPYD MARE/Lib
1	NAVSHIPYD NORVA/Lib
1	NAVSHIPYD PEARL/Lib
1	NAVSHIPYD PHILA/Lib
1	NAVSHIPYD PTSMH/Lib
12	DTIC
1	AFFDL/FDDS/J. Olsen
2	AFFDL/FYS 1 Dale Cooley 1 S.J. Pollock
2	COGARD 1 COM (E), STA 5-2 1 Div of Merchant Marine Safety
1	LC/Sci & Tech Div
1	MARAD/Adv Ship Prog Off
1	MMA/Tech Lib
1	NASA AMES RES CEN R.T. Medan, Ms 221-2
3	NASA LANGLEY RES CEN 1 J.E. Lamar, Ms 404A 1 Brooks 1 E.C. Yates, Jr., Ms 340
1	NASA Sci & Tech Info Facility
1	NSF/Eng Div

Copies		Copies	
1	Univ of Bridgeport Prof E. Uram Mech Eng Dept	1	Kansas State Univ Engineering Exp Station D.A. Nesmith
5	Univ of California, Berkeley College of Eng, NA Dept 1 Lib 1 J.R. Paulling 1 J.V. Wehausen 1 W. Webster 1 R. Yeung	1	Lehigh Univ/Fritz Lab Lib
		1	Long Island Univ Grad Dept of Marine Sci David Price
		1	Delaware Univ/Math Dept
3	Calif Inst of Tech 1 A.J. Acosta 1 T.Y. Wu 1 Library	4	Univ of Maryland 1 Eng Lib 1 P.F. Cunniff 1 C.L. Sayre 1 F. Buckley
1	Colorado State Univ M. Albertson Dept of Civ Eng	6	Mass Inst of Tech Dept of Ocean Eng 1 T.F. Ogilvie 1 J.R. Kerwin 1 N. Neumann 1 P. Leehey 1 M. Abkowitz 1 Hydro Lab
1	Univ of Connecticut V. Scottron Hyd Res Lab		
1	Cornell Univ Grad School of Aero Eng	2	Univ of Mich/Dept NAME 1 R. Beck 1 R.B. Couch
1	Florida Atlantic Univ Ocean Eng Lib		
1	Harvard Univ/Dept of Math G. Birkhoff	1	Univ of Minn/St. Anthony Falls 1 R. Arndt
1	Univ of Hawaii Dr. Bretschneider	3	City College, Wave Hill 1 W.J. Pierson, Jr. 1 A.S. Peters 1 J.J. Stoker
1	Univ of Illinois College of Eng J.M. Robertson Theoretical & Applied Mech	1	Univ of Notre Dame A.F. Strandhagen
3	State Univ of Iowa Iowa Inst of Hyd Res 1 L. Landweber 1 J. Kennedy 1 V.C. Patel	1	Penn State Univ Applied Research Lab
		3	Southwest Res Inst 1 H.N. Abramson 1 G.E. Transleben, Jr. 1 Applied Mech Review

Copies	
3	Stanford Univ/Dept of Civ Eng 1 R.L. Street 1 B. Perry 1 Dept of Aero and Astro/ J. Ashley
1	Stanford Res Inst/Lib
2	Stevens Inst of Tech/Davidson Lab 1 J.P. Breslin 1 Lib
1	Utah State Univ/Col of Eng Roland W. Jeppson
1	Univ of Virginia/Aero Eng Dept
2	Webb Institute 1 Library 1 L.W. Ward
1	Worcester Poly Inst/Alden Res Lab
1	Woods Hole, Ocean Eng Dept
1	SNAME
1	Aerojet-General/W.C. Beckwith
1	Bethlehem Steel Sparrows Tech Mgr
1	Bolt, Beranek & Newman, MA
3	Boeing Company 1 Aerospace Group 1 Commercial Airplane Group 1 Marine System Group
1	CALSPAN, Inc./Applied Mech Dept
1	Flow Research, Inc.
1	Eastern Res Group

Copies	
2	General Dynamics Corp 1 Convair Aerospace Div A.M. Cunningham, Jr. Ms 2851 1 Electric Boat Div V.T. Boatwright, Jr.
1	Gibbs & Cox, Inc. Tech Info Control Section
1	Grumman Aircraft Eng Corp W.P. Carl, Mgr. Grumman Marine
1	S.F. Hoerner
2	Hydronautics, Inc. 1 M.P. Tulin 1 R. Barr
4	Lockheed Aircraft Corp Lockheed Missiles & Space 1 R.L. Waid 1 R. Lacy 1 R. Perkins 1 R. Kramer
1	Marquadt Corp/F. Lane General Applied Sci Labs
1	Martin Marietta Corp/Rias Peter F. Jordan
3	McDonnell-Douglas Corp Douglas Aircraft Company 1 Lib 1 J. Hess 1 T. Cebeci
1	Newport News Shipbuilding/Lib
1	Nielsen, NA Rockwell
1	North American Rockwell Los Angeles Div J.R. Tulinius/Dept 056-015
2	Northrop Corp/Aircraft Div 1 J.T. Gallagher 1 J.R. Stevens

Copies	
1	Oceanics, Inc. Paul Kaplan
1	Sperry Sys Mgmt
1	Robert Taggart, Inc.
1	Tracor

CENTER DISTRIBUTION

Copies	Code	Name
1	012.2	B.V. Nakonechny
1	11	
1	115	
1	1151	
1	1152	
1	1154	
1	15	
1	1504	
1	1506	
1	152	
1	1521	
1	1522	
10	154	
2	1540.1	
1	1541	
3	1543	
	1	P.K. Besch
	1	E.P. Rood
	1	D. Coder
13	1552	
	10	T.T. Huang
	1	M.S. Chang
	1	K.J. Bai
	1	F. Noblesse
1	156	

Copies	Code	Name
1	1561	
1	1562	
2	1563	
2	1564	
1	16	
1	163	
1	166	
1	17	
1	18	
1	1802.1	
2	1843	
1	19	
1	273	
10	5211.1	Reports Distribution
1	522.1	Unclassified Lib (C)
1	522.2	Unclassified Lib (A)